DOCUMENT RESUME

ED 175 701 SE 028 689

AUTHOR Schaaf, William L., Ed.
TITLE Reprint Series: Infinity. RS-14.
INSTITUTION Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 69
NOTE 62p.: For related documents, see SE 028 676-690

EDRS PRICE MF01/PC03 Plus Postage.
DESCRIPTORS Curriculum; *Enrichment; *Instruction; *Mathematical Applications; Mathematics Education; *Number Concepts; *Numbers; Secondary Education; *Secondary School Mathematics; Supplementary Reading Materials
IDENTIFIERS *School Mathematics Study Group

ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series makes available expository articles which appeared in a variety of mathematical periodicals. Topics covered include: (1) is there an infinity; (2) infinity and its presentation at the high school level; (3) the hierarchy of infinities and the problems it spawns; and (4) the motionless arrow. (MP)

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which do not find a place in the curriculum simply because of lack of time, even though they are well within the grasp of secondary school students.

Some classes and many individual students, however, may find time to pursue mathematical topics of special interest to them. The School Mathematics Study Group is preparing pamphlets designed to make material for such study readily accessible. Some of the pamphlets deal with material found in the regular curriculum but in a more extended manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum.

This particular series of pamphlets, the Reprint Series, makes available expository articles which appeared in a variety of mathematical periodicals. Even if the periodicals were available to all schools, there is convenience in having articles on one topic collected and reprinted as is done here.

This series was prepared for the Panel on Supplementary Publications by Professor William L. Schaaf. His judgment, background, bibliographic skills, and editorial efficiency were major factors in the design and successful completion of the pamphlets.

PREFACE

In everyday conversation we are prone to use the word "infinite" so casually that we are scarcely aware of its subtle mathematical meanings. Thus we speak of *infinite patience*, or an *infinite hoard* of insects, or an *infinite variety* of designs, and so forth. Such usage may serve a useful emotional purpose or please our literary fancy, but it is hardly sound mathematics. To the uninitiated, the word *infinite* is generally associated with the idea of bigness — enormity. Yet nothing could be more misleading. The number of drops of water in all the oceans of the world, to be sure, is very great. But presumably if we counted them one by one, we would eventually come to the "last" drop. There is some number that tells "how many" drops there are. Although it would contain many, many digits, it would be a finite number.

Now consider the sequence of integers itself: 1, 2, 3, 4, · · · . Upon reaching any integer n, however great, there is always another integer that immediately follows it, namely, $n + 1$. Can this procedure of naming the successor of any chosen integer ever cease? Is there a "greatest" integer? Is there a "last" number? The answer is clearly, NO. But do not regard this idea lightly! It has been aptly said that the distinction between a sequence with a beginning and an end, and one that never ends, is literally awe-inspiring.

The key-word here is the word *endless*. A simple word, but an elusive idea. How can we visualize or think about something that "never ends"? Yet the fact remains that the human mind is capable of grasping something that it cannot literally experience. We cannot "see", let alone name, *all* the integers. But by a stroke of the wand we can say: "There they are — all of them at once." And then we can proceed to talk about the set of all the integers, or the set of all the points on a line segment.

The concept of the infinite not only baffled but challenged mankind for ages. The Greeks wrestled with the idea, as witness the famous paradoxes of Zeno. Throughout the succeeding centuries mathematicians continued to struggle with the concept, particularly from the time of

Newton and Leibniz on. Not until the end of the 19th century was there a significant break-through.

The concept of an infinite set of things is relatively modern. The father of this concept was Georg Cantor, who, about 1895, began to develop a theory of classes, or set theory (Mengenlehre). Building on this theory, he developed the doctrine of transfinite numbers, a doctrine which David Hilbert, another founder of modern twentieth-century mathematics, hailed as "one of the greatest achievements of human reason." It is chiefly this aspect of the infinite which is so ably discussed in the following essays.

— William L. Schaaf

CONTENTS

ACKNOWLEDGMENTS

The SCHOOL MATHEMATICS STUDY GROUP takes this opportunity to express its gratitude to the authors of these articles for their generosity in allowing their material to be reproduced in this manner and for this purpose: Professor Hans Hahn, the distinguished Austrian mathematician and philosopher; Margaret Willerding, who at the time her paper was published, was associated with the San Diego State College, San Diego, California; Martin Gardner, a well-known free-lance writer on a wide variety of mathematical subjects; and the late Professor Nathan Altshiller Court, of the Department of Mathematics and Astronomy, the University of Oklahoma.

The SCHOOL MATHEMATICS STUDY GROUP would like also at this time to express its sincere appreciation to the several editors and publishers who have graciously allowed these articles to be reprinted:

SCIENTIFIC AMERICAN

HANS HAHN, *"Is There an Infinity?,"* vol. 187, pp. 76–80 (November, 1952).

MARTIN GARDNER, *"The Hierarchy of Infinities and the Problems it Spawns,"* vol. 214, pp. 112–116 (March, 1966).

SCIENTIFIC MONTHLY

N. A. COURT, *"The Motionless Arrow,"* vol. 63, pp. 249–256 (1946).

SCHOOL SCIENCE AND MATHEMATICS

MARGARET F. WILLERDING, *"Infinity and its Presentation at the High School Level,"* vol. 63, pp. 463–474 (June, 1963).

FOREWORD

In the long slow evolution of mathematical thought, spanning at least five thousand years, several notable breakthroughs occurred at various times. One of the earliest was the Greek approach to demonstrative geometry, or deductive proof in geometry. Another was the marriage of algebra and geometry, or the invention of analytic geometry, by Descartes about 1637. In more recent times, we had the creation of non-Euclidean geometries by Lobachevski and others. Not all the significant milestones were in the field of geometry, of course. Some were in arithmetic, in algebra, in logic, and in other fields.

Less than a century ago one of the most far reaching and revolutionary breakthroughs was the theory of aggregates, or Mengenlehre, developed by Georg Cantor about 1880, and known today as set theory. Along with the theory of sets Cantor also created the transfinite numbers and the continuum. When he first announced his invention he wrote to a friend: "I see it, but I don't believe it!" Some of the implications of the theory were so startling, especially those concerning infinite sets, that not a few contemporary mathematicians were reluctant to accept the new doctrine. Indeed, despite the fact that the theory of transfinite numbers has since become quite respectable, there are still a few mathematicians who are somewhat skeptical about the existence of transfinite numbers and operations with such numbers.

Be that as it may, the concepts developed by Cantor and others in this field led to other highly significant mathematical results of indispensible service to science, philosophy and cosmology. In this first essay, the author, in bold strokes, gives a broad overview not only of Cantor's theory of the infinite, but also some of the implications for contemporary physics and relativity theory. In this connection we are reminded of a perceptive observation made by the late J. W. N. Sullivan, who expressed the conviction that

> "the significance of mathematics resides precisely in the fact that it is an art; by informing us of the nature of our own minds it informs us of much that depends on our minds. It does not enable us to explore some remote region of the externally existent; it helps to show us how far what exists depends upon the way we exist. We are the lawgivers of the universe; it is even possible that we can experience nothing but what we have created, and that the greatest of our mathematical creations is the material universe itself."

Is There an Infinity?

The great German mathematician Georg Cantor proved that, so far as mathematics is concerned, there is. Presenting a celebrated account of his ideas and their consequences.

Since ancient times philosophers, theologians and mathematicians have occupied themselves with the subject of infinity. Zeno of Elea invented a group of famous paradoxes whose difficulties are connected with the concept; in their time such leading thinkers as Aristotle, Descartes, Leibnitz and Gauss grappled with the infinity problem without making any notable contributions to its clarification. The subject is admittedly complex and undeniably important. A firm grasp of the problems of infinity is essential to an understanding of the revolution in ideas that paved the way for the triumphant advance of modern mathematics, with important consequences to physics, cosmology and related sciences.

The following article is a condensed version of a lecture on infinity by a noted Austrian mathematician, Hans Hahn, delivered a few years ago before a general audience in Vienna. This is the first translation of the lecture into English. Hahn was a member of the celebrated Vienna Circle, a group of philosophers and scientists adhering to the philosophy of logical positivism, among whose founders were Otto Neurath and Rudolf Carnap. The Circle annually presented popular lectures on science, and this survey by Hahn of the concept of infinity is one of the [illegible] of the series.

Hahn began his lecture with a historical résumé (here omitted) and then launched his discussion with a description of the work of the founder of the modern mathematical theory of infinity, Georg Cantor.

Hans Hahn

It was Georg Cantor who in the years 1871–84 created a completely new and very special mathematical discipline, the theory of sets, in which was founded, for the first time in a thousand years of argument back and forth, a theory of infinity with all the incisiveness of modern mathematics. Like so many other new creations this one began with a very simple idea. Cantor asked himself: "What do we mean when we say of two finite sets that they consist of equally many things, that they have the same number, that they are equivalent?" Obviously nothing more than this: that between the members of the first set and those of the second a correspondence can be effected by which each member of the first set matches exactly a member of the second set, and likewise each member of the second set matches one of the first. A correspondence of this kind is called "reciprocally unique," or simply "one-to-one." The set of the fingers of the right hand is equivalent to the set of fingers of the left hand, since between the fingers of the right hand and those of the left hand a one-to-one pairing is possible. Such a correspondence is obtained, for instance, when we place the thumb on the thumb, the index finger on the index finger, and so on. But the set of both ears and the set of the fingers of one hand are not equivalent, since in this instance a one-to-one correspondence is obviously impossible: for if we attempt to place the fingers of one hand in correspondence with our ears, no matter how we contrive there will necessarily be some fingers left over to which no ears correspond. Now the number (or cardinal number) of a set is obviously a characteristic that it has in common with all equivalent sets, and by which it distinguishes itself from every set not equivalent to itself. The number 5, for instance, is the characteristic which all sets equivalent to the set of the fingers of one hand have in common, and which distinguishes them from all other sets.

Thus we have the following definitions: Two sets are called equivalent if between their respective members a one-to-one correspondence is possible; and the characteristic that one set has in common with all equivalent sets, and by which it distinguishes itself from all other sets not equivalent to itself, is called the (cardinal) number of that set. And now we make the fundamental assertion that in these definitions the finiteness of the sets considered is in no sense involved; the definitions can be applied as readily to infinite sets as to finite sets. The concepts "equivalent" and "cardinal number" are thereby transferred to sets of infinitely many objects. The cardinal numbers of finite sets, *i.e.*, the numbers 1, 2, 3 . . . are called natural numbers; the cardinal numbers of infinite sets Cantor calls "transfinite cardinal numbers."

But are there really any infinite sets? We can convince ourselves of this at once by a very simple example. There are obviously infinitely many different natural numbers; hence the set of all the natural numbers contains infinitely many members: it is an infinite set. Now then, those sets that are equivalent to the set of all natural numbers, whose members can be paired in a one-to-one correspondence with the natural numbers, are called denumerably infinite sets. . . . According to our definition all denumerably infinite sets have the same cardinal number; this cardinal number must now be given a name, just as the cardinal number of the set of the fingers on one hand was earlier given the name 5. Cantor gave this cardinal number the name "aleph-null." written $\aleph_0$. (Why he gave it this rather bizarre name will become clear later.) The number $\aleph_0$ is thus the first example of a transfinite cardinal number. Just as the statement "a set has the number 5" means that its members can be put in one-to-one correspondence with the fingers of the right hand, or — what amounts to the same thing — with the integers 1, 2, 3, 4, 5, so the statement "a set has the cardinal number $\aleph_0$," means that its members can be put in one-to-one correspondence with the totality of natural numbers.

If we look about us for examples of denumerably infinite sets, we arrive immediately at some highly surprising results. The set of all natural numbers is itself denumerably infinite; this is self-evident, for it was from this set that we defined the concept "denumerably infinite." But the set of all even numbers is also denumerably infinite, and has the same cardinal number $\aleph_0$ as the set of all natural numbers, though we would be inclined to think that there are far fewer even numbers than natural numbers. To prove this proposition we have only to put each natural number opposite its double (*see below*). It may clearly be seen that there is a one-to-one correspondence between all natural and all even numbers, and thereby our point is established. In exactly the same way it can be shown that the set of all odd numbers is denumerably infinite.

Even more surprising, perhaps, is the fact that the set of all pairs of natural numbers is denumerably infinite. In order to understand this we have merely to arrange the set of all pairs of natural numbers diagonally, whereupon we at once obtain a one-to-one correspondence between all natural numbers and all pairs of natural numbers [*see table on page 8*]. From this follows the conclusion, which Cantor discovered while still a student, that the set of all rational fractions (*i.e.*, fractions in which the numerators and denominator are whole numbers, such as 1/2, 2/3, *etc.*) is also denumerably infinite, or equivalent to the set of all natural numbers, though again one might suppose that there are many, many more fractions than there are natural numbers. What is more, Cantor was able to prove that the set of all so-called algebraic numbers, that is, the set of all numbers that satisfy an algebraic equation of the form $a_0x^n + a_1x^{n-1} +$

$\ldots + a_{n-1}x + a^n = 0$ with integral coefficients $a_0, a_1 \ldots a_n$, is denumerably infinite.

At this point the reader may ask whether, in the last analysis, *all* infinite sets are not denumerably infinite – that is, equivalent. If this were so, we should be sadly disappointed; for then, alongside the finite sets there would simply be infinite ones which would all be equivalent, and there would be nothing more to say about the matter. But in the year 1874 Cantor succeeded in proving that there are also infinite sets that are not denumerable; that is to say, there are other infinite numbers, transfinite cardinal numbers, differing from aleph-null. Specifically, Cantor proved that the set of all so-called real numbers (*i.e.*, composed of all whole numbers, plus all fractions, plus all irrational numbers) is non-denumerably infinite.

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \\ 2 & 4 & 6 & 8 & 10 & 12 & & \ldots \end{array}$$

SET OF EVEN NUMBERS = ALL INTEGERS

[*The essence of Hahn's account of Cantor's proof is that no comprehensive counting procedure can be devised for the entire set of real numbers, nor even for one of its proper subsets, such as all the real numbers lying between 0 and 1. While the members of a specifically described infinite set, such as all rational fractions or all algebraic numbers, can be paired off with the natural numbers, every attempt to construct a formula for counting the all-inclusive set of real numbers is invariably frustrated. No matter what counting scheme is adopted, it can be shown that some of the real numbers in the set so considered remain uncounted, which is to say that the scheme fails. It follows that an infinite set for which no counting method can be devised is non-countable, in other words non-denumerably infinite.*]

It has thus been shown that the set of natural numbers and the set of real numbers are not equivalent; that these two sets have different cardinal numbers. The cardinal number of the set of real numbers Cantor called the "power of the continuum"; we shall designate it by *c*. Earlier it was noted that the set of all algebraic numbers is denumerably infinite, and we just now saw that the set of all real numbers is not denumerably infinite; hence there must be real numbers that are not algebraic. These are the so-called "transcendental" numbers, whose existence is demonstrated in the simplest way conceivable by Cantor's brilliant train of reasoning.

It is well known that the real numbers can be put in one-to-one correspondence with the points of a straight line; hence c is also the cardinal number of the set of all points of a straight line. Surprisingly Cantor was also able to prove that a one-to-one pairing is possible between the set of all points of a plane and the set of all points of a straight line. These two sets are thus equivalent; that is to say, c is also the cardinal number of the set of all points of a plane, though here too we should have thought that a plane would contain a great many more points than a straight line. In fact, as Cantor has shown, c is the cardinal number of *all* points of three dimensional space, or even of a space of any number of dimensions.

We have discovered two different transfinite cardinal numbers, $\aleph_0$ and c: the power of the denumerably infinite sets and the power of the continuum. Are there yet others? Yes, there certainly are infinitely many different transfinite cardinal numbers; for given any set M, a set with a higher cardinal number can at once be indicated, since the

CANTOR'S ALEPH-NULL

set of all possible subsets of M has a higher cardinal number than the set M itself. Take, for example, a set of three things, such as the set of the three figures 1, 2, 3. Its partial sets are the following: 1; 2; 3; 1, 2; 2, 3; 1, 3–thus the number of partial sets is more than three. Cantor has shown that this is generally true, even for infinite sets. For example, the set of all possible point-sets of a straight line has a higher cardinal number than the set of all points of the straight line; that is to say, its cardinal number is greater than c.

What is now desired is a general view of all possible transfinite cardinal numbers. As regards the cardinal numbers of finite sets, the natural numbers, the following simple situation prevails: Among such sets there is one that is the smallest, namely 1; and if a finite set M with the cardinal number m is given, a set with the next-larger cardinal number can be formed by adding one more object to the set M. What is the rule in this respect with regard to infinite sets? It can be shown without difficulty that among the transfinite cardinal numbers, as well as among the finite ones, there is one that is the smallest, namely $\aleph_0$, the power of denumerably infinite sets (though we must not think that this is self-evident, for among all positive fractions, for instance, there is none that is the smallest). It is, however, not so easy as it was in the case of finite sets to form the next-larger to a transfinite cardinal number; for whenever we add one more member to an infinite set we do not get a set of greater cardinality, only one of equal cardinality. But Cantor also solved this difficulty, by showing that there is a next-larger to every transfinite cardinal number . . . and by showing how it is obtained. We cannot go into his proof here, since this would take us too far into the realm of pure mathematics. It is enough for us to recognize the fact that there is a smallest transfinite cardinal number, namely $\aleph_0$; after this

$$
\begin{tikzcd}
(1,1) \arrow[d] & (1,2) \arrow[r] & (1,3) \arrow[dl] & (1,4) \arrow[r] & (1,5) \arrow[dl] & \cdots \\
(2,1) \arrow[ur] & (2,2) \arrow[dl] & (2,3) \arrow[ur] & (2,4) \arrow[dl] & \cdots & \\
(3,1) \arrow[d] & (3,2) \arrow[ur] & (3,3) \arrow[dl] & \cdots & & \\
(4,1) \arrow[ur] & (4,2) \arrow[dl] & \cdots & & & \\
(5,1) \arrow[d] & \cdots & & & & \\
\cdots & & & & &
\end{tikzcd}
$$

THE DENUMERABLE INFINITY OF PAIRS OF NUMBERS

there is a next-larger, which is called $\aleph_1$; after this there is again a next-larger, which is $\aleph_2$, and so on. But this still does not exhaust the class of transfinite cardinals; for if it be assumed that we have formed the cardinal numbers $\aleph_0, \aleph_1, \aleph_2 \ldots \aleph_{10} \ldots \aleph_{100} \ldots \aleph_{1000} \ldots$ that is, all alephs ($\aleph_n$) whose index n is a natural number, then there is again a first transfinite cardinal number larger than any of these—Cantor called it $\aleph_\omega$—and a next-larger successor $\aleph_{\omega+1}$, and so on and on.

The successive alephs formed in this manner represent all possible transfinite cardinal numbers, and hence the power c of the continuum must occur among them. The question is which aleph is the power of the continuum. This is the famous problem of the continuum. We already know that it cannot be $\aleph_0$, since the set of all real numbers is non-denumerably infinite, that is to say, not equivalent to the set of natural numbers. Cantor took $\aleph_1$ to be the power of the continuum. The question, however, remains open. . . .

On the basis of this rather sketchy description of the structure of the theory of sets, the answer to the question, "Is there an infinity?" appears to be an unqualified "Yes." There are not only, as Leibnitz had already asserted, infinite sets, but there are even what Leibnitz had denied, infinite numbers, and it can also be shown that one can operate with them, in a manner similar to that used for finite natural numbers.

So far we have dealt only with the question whether there are infinite sets and infinite numbers; but no less important, it would appear, is the question whether there are infinite extensions. This is usually phrased in the form: "Is space infinite?" Let us begin by treating this question also from a purely mathematical standpoint.

We must recognize at the outset that mathematics deals with very diverse kinds of space. Here, however, we are interested only in the so-called Riemann spaces, and in particular in the three-dimensional Riemann spaces. Their exact definition does not concern us; it is sufficient to make the point that such a Riemann space is a set of elements, or points, in which certain subsets, called "lines," are the objects of attention. By a process of calculation there can be assigned to every such line a positive number, called the length of the line, and among these lines there are certain ones of which every sufficiently small segment AB is shorter than every other line joining the points A, B. These lines are called the geodesics, or the straight lines of the space in question. Now it may be that in any particular Riemann space there are straight lines of arbitrarily great length; in that case we shall say that this space is of infinite extension. On the other hand, it may also be that in this particular Riemann space the length of all straight lines remains less than a fixed number; then we say that the space is of finite extension. Until the end of the 18th century only one kind of mathematical space was known, and hence it was simply called "space." This is the space whose geometry is taught in school and which we call Euclidean space, after the Greek mathematician Euclid, who was the first to develop the geometry of this space systematically. And from our definition above, this Euclidean space is of infinite extension.

There are, however, also three-dimensional Riemann spaces of finite extension; the best known of these are the so-called spherical spaces (and the closely related elliptical ones), which are three-dimensional analogues of a spherical surface. The surface of a sphere can be conceived as a two-dimensional Riemann space, whose geodesics, or "straight" lines, are arcs of great circles. (A great circle is a circle cut on the surface of a sphere by a plane passing through the center of the sphere, as for instance, the equator and the meridians of longitude on the earth.) If r is the radius of the sphere, then the full circumference of a great circle is $2\pi r$; that is to say, no great circle can be longer than $2\pi r$. Hence the sphere considered as two-dimensional Riemann space is a space of finite extension. With regard to three-dimensional spherical space the situation is fully analogous; this also is a space of finite extension. Nevertheless, it has no boundaries; one can keep walking along one of its straight lines without ever being stopped by a boundary of the space. After a finite time one simply comes back to the starting point, exactly as if one had kept moving farther and farther along a great circle of a spherical surface. In other words, we can make a circular tour of spherical space just as easily as we can make a circular tour of the earth.

Thus we see that in a mathematical sense there are spaces of infinite extension (*e.g.*, Euclidean space) and spaces of finite extension (*e.g.*, spherical and elliptical spaces). Yet this is not at all what most persons have in mind when they ask: "Is space infinite?" They are asking, rather: "Is the space in which our experience and in which physical events take place of finite or infinite extension?"

So long as no mathematical space other than Euclidean space was known, everyone naturally believed that the space of the physical world was Euclidean space infinitely extended. Kant, who explicitly formulated this view, held that the arrangement of our observations in Euclidean space was an intuitional necessity; the basic postulates of Euclidean geometry are synthetic, *a priori* judgments.

But when it was discovered that in a purely mathematical sense spaces other than Euclidean also "existed" (that is, led to no logical contradictions), men began to question the position that the space of the physical world must be Euclidean space. And the idea developed that it was a question of experience, that is, a question that must be decided by experiment, whether the space of the physical world was Euclidean or not. Gauss actually made such experiments. But after the work of Henri Poincaré, the great mathematician of the end of the 19th century, we know that the question expressed in this way has no meaning. To a considerable extent we have a free choice of the kind of mathematical space in which we arrange our observations. The question does not acquire meaning until it is decided how this arrangement is to be carried out. For the important thing about Riemann space is the manner in which each of its lines is assigned a length, that is, how lengths are measured in it. If we decide that measurements of length in the space of physical events shall be made in the way they have been made from earliest times, that is, by the application of "rigid" measuring rods, then there is meaning in the question whether the space of physical events, considered as a Riemann space, is Euclidean or non-Euclidean. And the same holds for the question whether it is of finite or infinite extension.

The answer that many perhaps are prompted to give, "Of course, by this method of measurement physical space becomes a mathematical space of infinite extension," would be somewhat too hasty. As background for a brief discussion of this problem we must first give a short and very simple statement of certain mathematical facts. Euclidean space is characterized by the fact that the sum of the three angles of a triangle in such space is 180 degrees. In spherical space the sum of the angles of every triangle is greater than 180 degrees, and the excess over 180 degrees is greater the larger the triangle is in relation to the sphere. In the surface of a sphere, the two-dimensional analogue of spherical space, this point is presented to us very clearly. On the surface of a sphere, as already mentioned, the counterpart of the straight-line triangle of spherical space is a triangle whose sides are arcs of great circles, and it is a well-known proposition of elementary geometry that the sum of the angles of a spherical triangle is greater than 180 degrees, and that the excess over 180 degrees is greater the larger the surface area of the triangle. If a further comparison be made of spherical triangles of equal area on spheres of different sizes, it may be seen at once that the excess of the sum of the angles over 180 degrees is greater the smaller the diameter of the sphere, which is to say, the greater the curvature of the sphere. This gave rise to the adoption of the following terminology (and here it is simply a matter of terminology, behind which nothing whatever secret is hidden): A mathematical space is called "curved" if there are triangles in it the sum of whose angles deviates from 180 degrees. It is "positively curved" if the sum of the angles of every triangle in it (as in elliptical and spherical spaces) is greater than 180 degrees, and "negatively curved" if the sum is less than 180 degrees —as is the case in the "hyperbolic" spaces discovered by Bolyai and Lobachevsky.

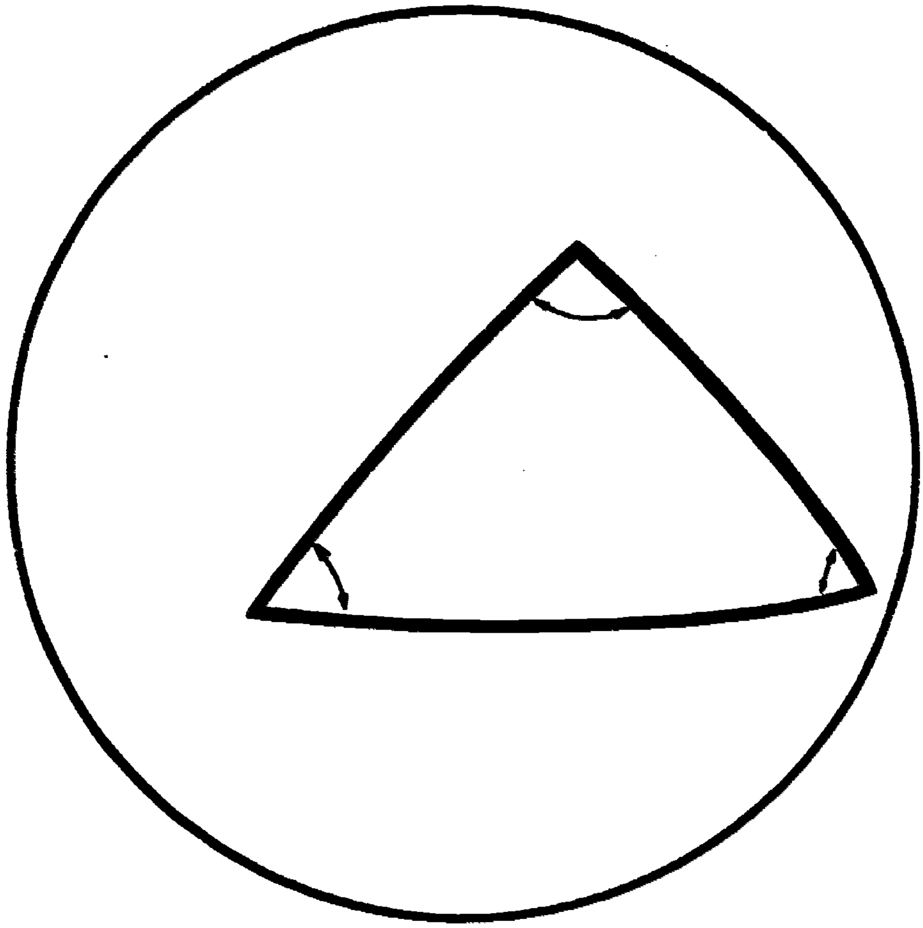
From the mathematical formulations of Einstein's General Theory of

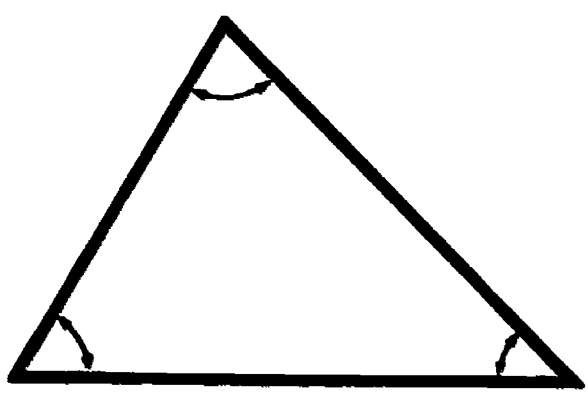
Relativity it now follows that, if the previously mentioned method of measurement is used as a basis, space in the vicinity of gravitating masses must be curved in a "gravitational field." The only gravitational field immediately accessible, that of the earth, is much too weak for us to be able to test this assertion directly. It has been possible, however, to prove it indirectly by the deflection of light rays—as determined during total eclipses—in the much stronger gravitational field of the sun. So far as our present experience goes, we can say that if, by using the measuring methods mentioned above, we turn the space of physical events into a mathematical Riemann space, this mathematical space will be curved, and its curvature will, in fact, vary from place to place, being greater in the vicinity of gravitating masses and smaller far from them.

To return to the question that concerns us: can we now say whether this space will be of finite or infinite extension? What has been said so far is not sufficient to give the answer; it is still necessary to make certain rather plausible assumptions. One such assumption is that matter is more or less evenly distributed throughout the entire space of the universe. The observations of astronomers to date can, at least with the help of a little good will, be brought into harmony with this assumption. Of course it can be true only when taken in the sense of a rough average, in somewhat the same sense as it can be said that a piece of ice has on the whole the same density throughout. Just as the mass of the ice is concentrated in a great many very small particles, separated by intervening spaces that are enormous in relation to the size of these particles, so the stars in world-space are separated by intervening spaces that are enormous in relation to the size of the stars. Let us make another quite plausible assumption, namely that by and large this average density of mass in the universe remains unchanged. We consider a piece of ice stationary, even though we know that the particles that constitute it are in active motion; we may likewise deem the universe to be stationary, even though we know the stars to be in active motion.

With these assumptions, then, it follows from the principles of the General Theory of Relativity that the mathematical space in which we are to interpret physical events must on the whole have the same curvature throughout. Such a space, however, like the surface of a sphere in two dimensions, is necessarily of finite extension. In other words, if we use as a basis the usual method of measuring length and wish to arrange physical events in a mathematical space, and if we make the two plausible assumptions mentioned above, the conclusion follows that this space must be of finite extension.

I said that the first of our assumptions, that of the equal density of





NON-EUCLIDEAN TRIANGLE (TOP) AND EUCLIDEAN TRIANGLE (BOTTOM) mass throughout space, conforms somewhat with observations. Is this also true of the second assumption, as to the constant density of mass with respect to time? Until recently this opinion was tenable. Now, however, certain astronomical observations seem to indicate—again speaking in broad terms—that all heavenly bodies are moving away from us with a velocity that increases the greater their distance from us, the velocity of those farthest away being quite fantastic. But if this is so, the average density of mass in the universe cannot possibly be constant in time; instead it must continually become smaller. Then if the remaining features of our picture of the universe are maintained, it would mean that we must assume that the mathematical space in which we interpret physical events is variable in time. At every instant it would be a space with (on the average) a constant positive curvature, that is to say, of finite extension, but the curvature would be continually decreasing while the extension would be continually increasing. This interpretation of physical events turns out to be entirely workable and in accord with the General Theory of Relativity.

But is this the only theory consistent with our experience to date? I said before that the assumption that the space of the universe was on the whole of uniform density could fairly well be brought into harmony with astronomical observations. At the same time, these observations do not contradict the entirely different assumption that we and our system of fixed stars are situated in a region of space where there is a strong concentration of mass, while at increasing distances from this region the distribution of mass keeps getting sparser. This would lead us — still using the ordinary method of measuring length — to conceive of the physical world as situated in a space whose curvature becomes smaller and smaller at increasing distances from our fixed star system. Such a space can be of infinite extension.

Similarly the phenomenon that the stars are in general receding from us, with greater velocity the farther away they are, can be quite simply explained as follows: Assume that at some time many masses with completely different velocities were concentrated in a relatively small region of space, let us say in a sphere K. In the course of time these masses will then, each with its own particular velocity, move out of this region of the space. After a sufficient time has elapsed, those that have the greatest velocities will have moved farthest away from the sphere K, those with lesser velocities will be nearer to K, and those with the lowest velocities will still be very close to K or even within K. Then an observer within K, or at least not too far removed from K, will see the very picture of the stellar world that we have described above. The masses will on the whole be moving away from him, and those farthest away will be moving with the greatest velocities. We would thus have an interpretation of the physical world in an entirely different kind of mathematical space — that is to say, in an infinitely extended space.

In summary we might very well say that the question, "Is the space of our physical world of infinite or of finite extension?" has no meaning as it stands. It does not become meaningful until we decide how we are to go about fitting the observed events of the physical world into a mathematical space, that is, what assumptions must be made and what logical requirements must be satisfied. And this in turn leads to the question, "Is a finite or an infinite mathematical space better adapted for the arrangement and interpretation of physical events?" At the present stage of our knowledge we cannot give any reasonably well-founded answer to this question. It appears that mathematical spaces of finite and of infinite extension are almost equally well suited for the interpretation of the observational data thus far accumulated.

Perhaps at this point confirmed "finitists" will say: "If this is so, we prefer the scheme based on a space of finite extension, since any theory incorporating the concept of infinity is wholly unacceptable to us." They are free to take this view if they wish, but they must not imagine thereby to have altogether rid themselves of infinity. For even the finitely extended Riemann spaces contain infinitely many points, and the mathematical treatment of time is such that each time-interval, however small, contains infinitely many time-points.

Must this necessarily be so? Are we in truth compelled to lay the scene of our experience in a mathematical space or in a mathematical time that consists of infinitely many points? I say no. In principle one might very well conceive of a physics in which there were only a finite number of space points and a finite number of time points — in the language of the theory of relativity, a finite number of "world points." In my opinion neither logic nor intuition nor experience can ever prove the impossibility of such a truly finite system of physics. It may be that the various theories of the atomic structure of matter, or today's quantum physics, are the first foreshadowings of a future finite physics. If it ever comes, then we shall have returned after a prodigious circular journey to one of the starting points of Western thought, that is, to the Pythagorean doctrine that everything in the world is governed by the natural numbers.

If the famous theorem of the right-angle triangle rightly bears the name of Pythagoras, then it was Pythagoras himself who shook the foundations of his doctrine that everything was governed by the natural numbers. For from the theorem of the right triangle there follows the existence of line segments that are incommensurable, that is, whose relationship with one another cannot be expressed by the natural numbers. And since no distinction was made between mathematical existence and physical existence, a finite physics appeared impossible. But if we are clear on the point that mathematical existence and physical existence mean basically different things; that physical existence can never follow from mathematical existence; that physical existence can in the last analysis be proved only by observation, and that the mathematical difference between rational and irrational forever transcends any possibility of observation — then we shall scarcely be able to deny the possibility in principle of a finite physics. Be that as it may, whether the future produces a finite physics or not, there will remain unimpaired the possibility and the grand beauty of a logic and a mathematics of the infinite.

FOREWORD

Although this essay has a slightly pedagogical flavor, it is nevertheless a very clear, concise and intelligible account of the general concept of infinity. Where the preceding essay may have stretched your imagination you will find this article somewhat easier reading. It will also suggest aspects of algebra with which you may very well be familiar. At all events, it is likely to serve a double purpose: it may clear up some points in Professor Hahn's article that you did not understand, and it may prepare you to appreciate more fully the following article by Mr. Gardner.

Infinity and Its Presentation at the High School Level

Margaret F. Willerding

Introduction

The nature of this paper is twofold; first, it is a 'popular' account of the work of Georg Cantor on infinity, and second, it illustrates a method of presenting some of his theories, and other related ones, to a student at the high school level.

Although Georg Cantor succeeded in defining infinity in mathematical terms about eighty years ago, his success has not yet penetrated to our high school teaching (nor, it would seem, to many of the twentieth century philosophers who still regard infinity as a paradox).

It is not to be supposed that there are no problems left concerning the infinite, nor, that the new definition of infinity has not engendered some of its own paradoxes; there is still a large field for further investigation. However, most mathematicians accept the work of Cantor as proof of the existence of infinity.

It is hard to understand the reluctance of educators to put some of this important material into the high school curriculum. It is not difficult and it is very important. The purpose of this paper is to explain just what infinity is all about, how it is taught in schools today and how it should be taught. It would be going too far afield to attempt a complete historical treatise on infinity or an analysis of all the philosophical and mathematical subtleties that have arisen in the wake of Cantor's work, but some will be mentioned.

In defining infinity it will be found necessary to talk about the general concept of number and give it a precise definition. It will also be necessary to dispel some 'common-sense' notions about both number and classes.

The concepts of number, cardinal number, infinite set, and one-to-one correspondence are extremely important. If this paper can give an insight into some of these concepts, as well as give a historical account of Cantor's efforts, its purpose will have been realized.

Zeno's Paradoxes

Philosophers have puzzled over infinity ever since the ancient Greek world of Socrates and Plato. Zeno, who appears in Plato's *Parmenides* as Socrates' instructor, invented a number of ingenious arguments concerning the infinitesimal, continuity and the infinite. It is difficult to know exactly what Zeno's words really were since we know of his arguments only through Aristotle, who was criticizing Plato. It may be that Aristotle distorted the arguments in order to refute them.[1]

We shall consider briefly the three most famous paradoxes. It is these arguments that have led to most of the controversy and paradoxes concerning infinity for the last 2500 years. The first argument is that of the race-course. According to Burnet it is as follows:

> You cannot get to the end of a race-course. You cannot traverse an infinite number of points in a finite time. You must traverse the half of any given distance before you traverse the whole, and the half of that again before you can traverse it. This goes on ad infinitum, so that there are an infinite number of points in any given space, and you cannot touch an infinite number one by one in a finite time.[2]

His second argument concerns the race between Achilles and the tortoise. Again quoting Burnet:

> Achilles will never overtake the tortoise. He must first reach the place from which the tortoise started. By that time the tortoise will have got some way ahead. Achilles must then make up that, and again the tortoise will be ahead. He is always coming nearer, but he never makes up to it.[3]

In this argument we see that, for every position that Achilles is at, the tortoise is at another one. Thus both Achilles and the tortoise must occupy exactly the same number of positions or instants. But if Achilles actually overtakes the tortoise then he clearly must occupy all those positions that the tortoise occupies and more besides. Since this implies a contradiction then Achilles never can catch the tortoise. This argument is not so easily disposed of as the first one as we shall see.

Zeno's third argument consists in showing that an arrow in flight cannot move. He seems to be assuming that a given length of time has a finite number of instants. Thus at any given instant of time the arrow is

[1] Russell, Bertrand, *Our Knowledge of the External World* (W. W. Norton and Co., Inc., 1929), p. 183.

[2] Burnet, John, *Early Greek Philosophy* (London: A. and C. Black), p. 367.

[3] *Ibid.*

just exactly where it is and no other. It cannot be moving during the instant because this would put it in different places during the instant and require the instant to be divisible. Thus the arrow is in one position at one instant but miraculously in another position in the next instant although at no time is it ever moving! Therefore the arrow is at rest.

The first argument is disposed of easily. Zeno's assumption that adding an infinite number of terms always gives infinity for an answer is wrong. Consider the infinite series:

$$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + \ldots$$

Adding term by term we get, successively, 1/2, 3/4, 7/8, 15/16, 31/32, ... Obviously the sum of this infinite series never even gets as large as 1. Still, it requires some knowledge of infinity to prove this.

The second argument, however, forces us to rid ourselves of one of the notions that is deep-seated in all of us—that the whole is always greater than any of its parts. We shall return to this later after we have defined an infinite set.

The third argument, strangely enough, is actually true, though with modifications. The arrow really is at rest at every instant, though this does not lead to the conclusion that it is therefore not in motion. The solution of this paradox will also become apparent.

It seems that, in any case, we must define what we mean by an infinity of numbers, or an infinite set, in order to dispense with Zeno's paradoxes (and any others that have to do with infinity). This is precisely what Cantor set out to do.

Cantor's Theory of Transfinite Numbers

Cantor began with what we know of finite sets. He first had to decide what was meant by saying that two finite sets of objects have the same number of objects. At first glance it appears that we should simply count each set and compare our answers, but this just shifts the problem to another one. For, if we count the objects in the first set and end at 64 (say), and similarly arrive 64 for the second set, why do we say that the sets are numerically equal? Because 64 equals 64? But '64' is just the counting number we arrived at for the last object. The crux of the matter is that, in our counting system, the last number arrived at in counting *also* tells just how many objects have been counted. Our two sets are numerically equal, not because '64' is the last number arrived at in counting each set, but because there are exactly 64 objects in each set.

This is not sophistry. If we counted by using the names in a telephone directory beginning with the A's, arriving at the name Ackerman for our last object might be correct, but we still wouldn't know how many objects we had. The problem would become hopelessly snarled if we used different directories from different cities. Yet our counting would, in a sense, be correct. Of course we'd never know where we were, especially when we got to the 'Smiths'!

Now, however, we have gotten a glimpse of what counting is all about. It is really a correspondence between symbols (numbers) and objects. Two finite sets of objects may be said to have the same number of things if to each member of the first set there corresponds one and only one member of the second set and to each member of the second set there corresponds one and only one member of the first set. Such a correspondence is called a 'one-to-one' correspondence.

There are just as many fingers on our two hands as there are fingers on a pair of gloves, because, when we put the gloves on, no fingers of our hands are left sticking out and no fingers of the glove remain unfilled. Similarly, it is easy to tell if there are the same number of people in an auditorium as there are chairs by simply asking everyone to be seated. If there are no people left standing and no empty seats, there must be just as many chairs as people. Note particularly that we may not even know how many people or how many chairs there actually are, yet we still know that both quantities are exactly equal.

This becomes even more important when it would be impossible to count the number of objects in a set. Thus, if we exclude bigamists, there are exactly as many husbands as wives in the world although it would be impossible to count how many of each there are. It is just this one-to-one correspondence that we make use of when we count the members of a set. We let each object correspond to a number and vice-versa. Thus 'one' corresponds to the first object, 'two' to the second, 'three' to the third and so on. If there are sixty-four objects, the last object will correspond to the number 'sixty-four.' But since we know that there are just sixty-four numbers from 'one' to 'sixty-four,' then there are exactly sixty-four objects that we have counted.

The trouble with counting is that it becomes virtually impossible when dealing with extremely large numbers. Besides, counting requires that things be put in order so that there is a first object, a second object, a third object, etc. We saw earlier that this wasn't necessary to determine the equality of some sets, e.g., the number of husbands and wives in the world. We have,therefore, two kinds of numbers. ordinal numbers and cardinal numbers. Ordinal numbers tell 'which one.' Cardinal numbers tell 'how many.' Thus we say that Saturday is the seventh day of the week, but that there are seven days in a week. The first is an ordinal number and depends on the position of Saturday with respect to the other days; the second is a cardinal number which is not affected by position or order.

A cardinal number may be defined as follows: There are just as many fingers on my right hand as on my left. There are just as many letters in the word 'right' as there are fingers on my right hand. There are just as many natural numbers from one to five as fingers on my right hand. In fact, there are many classes of things which have the same number of objects as there are fingers on my right hand. All of these may be ascertained by our method of one-to-one correspondence. Each of these classes is called 'equivalent' to our original class. All of these classes have something in common, a certain 'fiveness,' and it is to this property that we give the cardinal number, five.

Now if we happen to know the cardinal number of one collection of objects and there exists a one-to-one correspondence between the objects of this collection and those of a second collection, then the second collection has the same cardinal number as the first. This follows directly from our definitions of equivalence and cardinal number.

We use this principle in daily life constantly without realizing it. If someone were to ask you how many squares there are in each row of a chess board, you would count one row and announce your answer—eight. If your interrogator were mathematically minded he might point out that you had only counted one row, not all of them. It is unlikely that you would try to prove there are eight squares in every row by then counting the other seven rows. More likely you would retort. "It's obvious. Every square in each row is set exactly below one square in the row above it. Therefore, since I counted eight in the first row, there are eight in every row." Here you have made use of the principle pointed out above, albeit unknowingly.

Again, every time you wish to find the number of objects in a rectangular array, you count the objects in one row and the objects in one column and multiply. That the answer you obtain is the correct one is only afforded by the truth of our principle (and the truth of the multiplication table).

Up to this point all is well. Cantor then went on to ask himself, "Why limit ourselves to finite sets; why not extend our definition to the infinite?" And so he did. The terms equivalent and cardinal number are used with no change in meaning to define infinite sets.

The most obvious example of an infinite set is the number of natural, or counting, numbers, i.e., the number of numbers beginning 1, 2, 3. . . . . Since these numbers never end (there is no largest number since one can always be added to it making it larger), there can be no ordinary cardinal number to tell how many there are. Whatever number it is, it is not finite; Cantor therefore called such numbers 'transfinite numbers'.[4] The cardinal number for the number of natural numbers he called 'aleph-null', written $\aleph_0$,[5] the first letter of the Hebrew alphabet. The subscript 0 indicates that this is the smallest (first) transfinite number. Any set equivalent to this set would have the same cardinal number $\aleph_0$. Such sets are called 'denumerably infinite'. We can now define how many natural numbers there are; there are exactly $\aleph_0$ of them.

Let us proceed to find some other denumerably infinite sets. We immediately find that the set of even numbers is also denumerably infinite. That is, there are just as many even numbers as there are natural numbers, even though the natural numbers contain all the even numbers and the odd ones as well. This seems impossible until we ask ourselves again what we mean by equality. Can we put the even numbers in a one-to-one correspondence with the natural numbers? The answer is 'yes' as seen in the following:

$$\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots & n & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & \ldots & 2n & \ldots \end{array}$$

To every natural number in the first row corresponds its double in the second row, and to every even number in the second row corresponds its half in the first row. Since every natural number can be doubled to give an even number and every even number halved to give a natural number, both rows are complete; that is, no integers and no even numbers have been left out or used more than once. It follows that the cardinality of the two sets is the same and they are equivalent. By a similar correspondence we find that there are just as many odd numbers as counting numbers.

It was just this example that caused Leibniz to aver that there were no infinite numbers. He claimed (correctly) that the above illustration showed that the whole is equal to one of its parts. Since this implied a

[4] Cantor, Georg, *Contributions to the Founding of the Theory of Transfinite Numbers* (New York: Dover Publications, Inc.), p. 103.

[5] *Ibid.*, p. 104.

contradiction, Leibniz rejected it.[4] But it is just this fact that Cantor (and, independently, Dedekind) used to define an infinite set. An infinite set is one which can be put into a one-to-one correspondence with a proper subset of itself.

Let us see if there are more denumerably infinite sets. We might expect that the number of fractions would be greater than the number of natural numbers. Considering all of the fractions there must surely be more of them than there are integers. But, by this time, we have found our common sense to be unreliable. Our problem is to attempt to set up a one-to-one correspondence between the fractions and the integers. This cannot be done by magnitude for there is no smallest fraction; neither is there, given any fraction, a next larger fraction. How then can we arrange the fractions to insure that we include all of them?

Cantor set up the following scheme: All fractions are really just a pair of integers, the numerator and the denominator. Since the sum of two integers is another integer, this affords us a method of arranging them. We shall take all fractions whose numerator and denominator add up to two, then those that add up to three, etc., as follows:

$$
\begin{array}{ccccccccccccc}
1/1 & & 1/2 & \to & 1/3 & & 1/4 & \to & 1/5 & & 1/6 & & \cdots \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & \swarrow & & \nearrow & & & \\
2/1 & & 2/2 & & 2/3 & & 2/4 & & 2/5 & & \cdots & & \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & & & & & \\
3/1 & & 3/2 & & 3/3 & & 3/4 & & \cdots & & & & \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & & & & & & \\
4/1 & & 4/2 & & 4/3 & & \cdots & & & & & & \\
 & \swarrow & & \nearrow & & & & & & & & & \\
5/1 & & 5/2 & & \cdots & & & & & & & & \\
\downarrow & \nearrow & & & & & & & & & & & \\
6/1 & & \cdots & & & & & & & & & &
\end{array}
$$

where the fractions are taken in the order indicated by the arrows. We see that no fractions will be left out by this scheme, since the first row contains all fractions whose numerator is 1, the second row all fractions whose numerator is 2, etc. Thus a one-to-one correspondence is set up as follows:

[4] Russell, Bertrand, *Our Knowledge of the External World* (New York: W. W. Norton and Company, Inc., 1929), pp. 207–208.

1	2	3	4	5	6	7	8	9	10	11
↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕
1/1	2/1	1/2	1/3	2/2	3/1	4/1	3/2	2/3	1/4	1/5
14	15	16	17	18	19	20	21	...		
↕	↕	↕	↕	↕	↕	↕	↕			
4/2	5/1	6/1	5/2	4/3	3/4	2/5	1/6	...		

We have shown that there are just as many fractions as integers.[7]

We also can see that adding 1 to every integer gives us a new set that has just as many numbers as before, as the following correspondence shows:

1	2	3	4	5	6	7	...
↕	↕	↕	↕	↕	↕	↕	
2	3	4	5	6	7	8	...

We are thus led to some strange arithmetic:

$$\aleph_0 + 1 = \aleph_0$$
$$\aleph_0 + 2 = \aleph_0$$
$$\vdots$$
$$\aleph_0 + \aleph_0 = \aleph_0$$

and,

$$2\aleph_0 = \aleph_0$$
$$3\aleph_0 = \aleph_0$$
$$\aleph_0 \cdot \aleph_0 = \aleph_0$$

This last equation results from our work with fractions. It is easily shown that n numbers taken two at a time form n^2 pairs.* Thus there are $\aleph_0^2$ fractions. But we showed that there were also $\aleph_0$ fractions, from which,

$\aleph_0^2 = \aleph_0$ follows.

Transfinite numbers lead to strange answers when we try to subtract or divide them. Substracting a finite number from $\aleph_0$ always gives $\aleph_0$ again, but what is the result of subtracting $\aleph_0$ from $\aleph_0$?†

[7] Hahn, Hans, "Infinity," *The World of Mathematics* (edit. by James R. Newman) (New York: Simon and Schuster, 1956), p. 1595.

* Strictly speaking, n^2 ordered pairs. (Editor)

† It can be shown that $\aleph_0 + n = \aleph_0$ on the basis of a rigorous definition of a cardinal sum: for two disjoint sets A and B, $N(A) + N(B) = N(A \cup B)$. Cf. Yarnelle, *An Introduction to Transfinite Mathematics*, pp. 18–24; Heath (1964). (Editor)

It equals any number from 0 to $\aleph_0$ as we shall show. Suppose from the set of all of the natural numbers we subtract the set of all the natural numbers. We will have nothing left and may write,

$$\aleph_0 - \aleph_0 = 0$$

If, now, from the counting numbers we subtract all integers greater than some number, say 12, the first twelve integers are left and we write,

$$\aleph_0 - \aleph_0 = 12$$

Now subtracting all the odd numbers from the counting numbers, we are left with all the even numbers, an infinite number. Thus

$$\aleph_0 - \aleph_0 = \aleph_0$$

It would seem at this point that all infinite collections have the same cardinal number $\aleph_0$, for, can't all infinite sets be put into some kind of one-to-one correspondence with the natural numbers? Cantor wondered about this, especially in regard to the number of real numbers, both rational and irrational. To make it easier, he considered only those real numbers between 0 and 1. Now it is known that every real number can be written as an infinite decimal expansion, repeating or non-repeating, depending on whether it is rational or irrational.‡ Thus,

$$1/3 = 0.3333333\ldots$$
$$1/7 = 0.142857142857\ldots$$
$$\sqrt{2}/2 = 0.707109\ldots$$
$$\pi/4 = 0.785398163\ldots$$

Cantor's problem was, as in the case of the fractions, to find some kind of scheme in which these decimals could be paired with the integers. Not discovering any, he assumed that a scheme had been found and then tried to find some real number which had possibly been omitted. Such a scheme would look something like this:

0.3154026 . . .
0.9684459 . . .
0.1243867 . . .
0.6864901 . . .
.
.

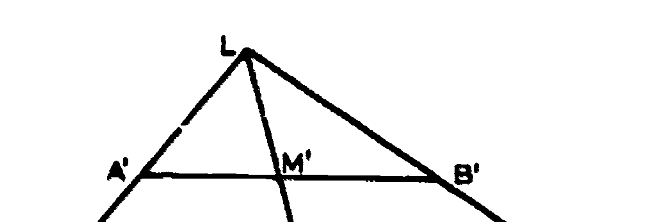
Now if there is any number which is not included in this array it

‡ The reader should note that .1000 . . . = .0999. . . . (Editor)

follows that it must differ from every single number in the array, in at least one digit. Cantor had to find just such a number. This he found could be done. He chose a number whose first digit differs from the first digit of the decimal in the first row, say 5; its second digit differs from the second digit of the decimal in the second row, say 3; its third digit differs from the third digit of the decimal in the third row, say 1, and so on. In this manner, the new number we have written will differ from every single decimal in the array. There is no use saying that this new number will appear 'somewhere' in the infinite array, for the question then is where? Is it in the 237th row? No, because our new number differs from the 237th number in its 237th digit, and similarly for any other position in the array. Now, since the number we made up was an arbitrary one, it is easy to see that any one of an infinite number of numbers could be written which would not appear in the array. Thus there are more real numbers than natural numbers. This new transfinite cardinal number Cantor called c, the 'power of the continuum'.[8] It is thought that c is probably $\aleph_1$, the next larger transfinite numbers after $\aleph_0$, but this has not been proved. Cantor also showed that there is an infinity of transfinite numbers, $\aleph_0$, $\aleph_1$, $\aleph_2$, . . . , but neither these nor the question of whether c and $\aleph_1$ are the same cardinal number need concern us here.

The number c is important, however, because it is known that the points on a line may be put in a one-to-one correspondence with the real numbers. Thus c is the cardinal number of the number of points on a line. Note also that the length of the line is unimportant. There are just as many points on a short line as on a long one! The following geometric diagram makes this apparent: (taken from Kasner and Newman).[9]

Let line AB be twice as long as $A'B'$. Let L be the intersection of



[8] Cantor, *op. cit.* p. 96.

[9] Kasner, Edward and Newman, James, *Mathematics and the Imagination* (New York: Simon and Schuster, 1940), p. 55.

lines AA' and BB'. A one-to-one correspondence can be made between the points of AB and those of $A'B'$ by drawing straight lines from L intersecting AB and $A'B'$ in corresponding points. Thus M corresponds to M' in the diagram. It is easily seen that for every point on $A'B'$ there corresponds a unique point on AB and vice-versa.

We may now return to Achilles who is still trying to overtake the tortoise. We claim that Achilles overtakes the tortoise and passes him, as common sense would suppose. Zeno would argue that this would require Archilles to be in more places than the tortoise but this is not so. We have just seen that a short line has the same number of points on it as a longer one. Thus, although Achilles travels a greater distance than the tortoise, they both touch the same number of points.

The arrow in flight may now be re-examined. We claimed that the arrow really is at rest at every instant. But, contrary to Zeno, time is infinitely divisible just as space is. The arrow is at one place one instant and at another place at another instant. Where is it at intermediate times? It is at intermediate points. There is no 'next' point to any given one. This may be difficult to accept, but only because we have fixed ideas about what we think motion really is.

At this point we will close our discussion of Cantor's work on infinity. For those who desire a more rigorous mathematical treatment of the ideas expounded here, there is no better reference than Cantor's original work, *Contributions to the Founding of the Theory of Transfinite Numbers*, with a long and valuable preface written by Philip E. B. Jourdain. This work is too abstruse to be covered in this paper, but it is excellent for anyone wishing to delve into the mathematical and logical developments of the theory of infinite sets.

Infinity in the High Schools

From what has gone before, it is evident that Cantor's theory can be taught in high school. It requires only a cleansing of the brain of some 'common-sense' notions for its truth and power to develop. Its importance cannot be denied. Even if the concept of the one-to-one correspondence were the only idea gotten across, this alone might justify its place in the curriculum. But more, without a thorough understanding of infinity, the student cannot be expected to grasp the full meaning of some of the most important parts of mathematics, e.g., set theory algebra, the theories of calculus, of continuity, and of limits. It is readily apparent that the theory of the infinite as discussed in this paper could be taught in not more than a week's time.

Let us examine very briefly just what is being taught in schools today. Although the infinite crops up in a number of instances in algebra and geometry, it is usually passed off by the teacher with little or no explanation at all.

It is only when geometric progressions are taught that infinity is finally admitted into the curriculum, and even then it is almost apologetically. Students are taught that the sum S of a geometric progression of n terms is given by the formula,

$$S = (a - ar^n)/(1 - r)$$

where a is the first of the progression and r the common ratio of one term to its predecessor. When r is a ratio less than 1, and n is allowed to become large without limit, the problem reduces itself to the following:

What is the value of ar^n as n becomes infinite? We may neglect the constant a, and write r as a fraction less than 1, e.g., $1/10$. Then, what is the value of $(1/10)^n$ as n becomes infinite? By allowing n to take on large, but *finite*, values it becomes obvious that $(1/10)^n$ becomes smaller and smaller, approaching 0. When n becomes infinite we say that $(1/10)^n$ equals 0. The formula for the progression under these circumstances becomes

$$S = a/(1 - r)$$

Although this is not quite rigorous and demands the concept of a limit, the answer is correct and the student's intuitive ideas suffice in this case. But, while this *application* of infinity is essentially correct, it leads to a complete misconception of the nature of infinity itself. The student notes that a big number, say 10 billion, can be used in this example to approximate infinity and this is the concept he keeps, i.e., that infinity is nothing more than a 'big number.' It is just this kind of intuitive thinking that we are trying to combat. If infinity were taught correctly, this idea of its being a big finite number would be dispelled. (It should be added here that the infinity that results from such examples as dividing 1 by 0 is not exactly what we have been discussing. We are concerned with the idea of infinite sets. But, since the student isn't taught either one of these ideas, this differentiation can be made later.)

I have examined a number of high school text-books; one of them is a so-called 'modern' edition of algebra (not to be confused with those algebra books that teach 'modern' algebra). None of these books gives any more to the student than the 'intuitive' idea of infinity.

Even one of the best new books, *Principles of Mathematics*, by Allendoerfer and Oakley, although an excellent book in most respects, completely misses the boat in its treatment of infinity. The one-to-one correspondence is mentioned in passing, and, as an example, the student is even asked to show that the even numbers 2, 4, 6, 8, . . . can be paired with the integers 1, 2, 3, 4, . . . but no particular significance is attached to the result, not even that it is unusual, or contrary to what we might expect. It seems incredible that this opportunity was missed. The whole concept of cardinal number, and of number itself, is also omitted, although there is a small reference made to *What Is Mathematics?* by Courant and Robbins.

As a brief summary we may ask ourselves why this subject is omitted and neglected. It would seem to be mostly a matter of inertia. Possibly with the new fields of modern mathematics that are opening up, we may expect even the educators to take more notice.

As a matter of solace, we may take refuge in knowing that until just recently, all of the mathematics taught in school had been known since the time of Newton, about 1700.

Bibliography

ALLENDOERFER, C. B. AND OAKLEY, C. O. *Principles of Mathematics*. New York, Toronto, London: McGraw-Hill Book Co., Inc., 1955.

BELL, E. T. *The Magic of Numbers*. New York, London: Whittlesey House, McGraw-Hill Book Co., Inc. 1946.

BURNETT, JOHN. *Early Greek Philosophy*, 4th edit. London: A. and C. Black. 1930.

CANTOR, GEORG. *Contributions to the Founding of the Theory of Transfinite Numbers*. New York: Dover Publications, Inc.

FREILICH, SHANHOLT, AND McCORMACK. *Intermediate Algebra*. New York, Chicago, and San Francisco: Silver Burdett Co., 1941.

HAHN, HANS. "Infinity." *The World of Mathematics*, Vol. III, (edit. by James R. Newman). New York: Simon and Schuster, 1956.

KASNER, EDWARD AND NEWMAN, JAMES R. *Mathematics and the Imagination*. New York: Simon and Schuster, 1940.

MAYOR AND WILCOX. *Algebra, Second Course*. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1957.

RUSSELL, BERTRAND. *Introduction to Mathematical Philosophy*. London: George Allen and Unwin, Ltd., 1919.

———. "Mathematics and the Metaphysicians." *Mysticism and Logic*. New York: W. W. Norton and Co., Inc., 1929.

———. *Our Knowledge of the External World*. New York: W. W. Norton and Co., Inc., 1929.

FOREWORD

Ever since Cantor's transfinite numbers first saw the light of day, mathematicians have speculated as to whether there was an infinity of greater power than aleph-null but of lesser power than the cardinal number C. It began to seem as if, like Goldbach's conjecture, this question might never be answered. After a lapse of some eighty-five years, it now appears that there is an answer—but a completely unexpected and somewhat disconcerting answer. We shall not deprive the reader of the pleasure of the surprise ending.

However, we shall take this opportunity to make an observation about the nature of mathematics, an observation which will be appreciated more fully after reading Mr. Gardner's suggestive essay. It is simply this: that the mathematics of the twentieth century is very different from the mathematics of preceding centuries, and it is quite possible that the mathematics of the twenty-first century may be so different from our present mathematics that we would scarcely call it mathematics at all. No one can foretell. Indeed, we may have to abandon the idea of a single unified mathematics altogether and be willing to accept several unrelated mathematics. What would they be like? We cannot possibly imagine.

The Hierarchy of Infinities and the Problems it Spawns

Martin Gardner

A graduate student at Trinity
Computed the square of infinity.
But it gave him the fidgets
To put down the digits,
So he dropped math and took up divinity.

—Anonymous

In 1963 Paul J. Cohen, a 29-year-old mathematician at Stanford University, found a surprising answer to one of the great problems of modern set theory: Is there an order of infinity higher than the number of integers but lower than the number of points on a line? To make clear exactly what Cohen proved, something must first be said about those two lowest known levels of infinity.

It was Georg Ferdinand Ludwig Philipp Cantor who first discovered that beyond the infinity of the integers—an infinity to which he gave the name aleph-null—there are not only higher infinities but also an infinite number of them. Leading mathematicians were sharply divided in their reactions. Henri Poincaré called Cantorism a disease from which mathematics would have to recover, and Hermann Weyl spoke of Cantor's hierarchy of alephs as "fog on fog."

On the other hand, David Hilbert said, "From the paradise created for us by Cantor, no one will drive us out," and Bertrand Russell once praised Cantor's achievement as "probably the greatest of which the age can boast." Today only mathematicians of the intuitionist school and a few philosophers are still uneasy about the alephs. Most mathematicians long ago lost their fear of them, and the proofs by which Cantor established his "terrible dynasties" (as they have been called by the Argentine writer Jorge Luis Borges) are now universally honored as being among the most brilliant and beautiful in the history of mathematics.

Any infinite set of things that can be counted 1, 2, 3 . . . has the cardinal number $\aleph_0$ (aleph-null), the bottom rung of Cantor's aleph ladder. Of course, it is not possible actually to count such a set; one merely shows how it can be put into one-to-one correspondence with the counting numbers. Consider, for example, the infinite set of primes. It is easily put in one-to-one correspondence with the positive integers:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 2 & 3 & 5 & 7 & 11 & 13 & \ldots \end{array}$$

The set of primes is therefore an aleph-null set. It is said to be "countable" or "denumerable." Here we encounter a basic paradox of all infinite sets. Unlike finite sets, they can be put in one-to-one correspondence with a *part* of themselves or, more technically, with one of their "proper subsets." Although the primes are only a small portion of the positive integers, as a completed set they have the same aleph number. Similarly, the integers are only a small portion of the rational numbers (the integers plus all integral fractions), but the rationals form an aleph-null set too.

There are all kinds of ways in which this can be proved by arranging the rationals in a countable order. The most familiar way is to attach them, as fractions, to an infinite square array of lattice points and then count the points by following a zigzag path, or a spiral path if the lattice includes the negative rationals. Here is another intriguing method of ordering and counting the positive rationals that was proposed by the American logician Charles Sanders Peirce.

Start with the fractions 0/1 and 1/0. (The second fraction is meaningless, but that can be ignored.) Sum the two numerators and then the two denominators to get the new fraction 1/1, and place it between the previous pair: 0/1, 1/1, 1/0. Repeat this procedure with each pair of adjacent fractions to obtain two new fractions that go between them:

$$\frac{0}{1} \quad \frac{1}{2} \quad \frac{1}{1} \quad \frac{2}{1} \quad \frac{1}{0}.$$

The five fractions grow, by the same procedure, to nine:

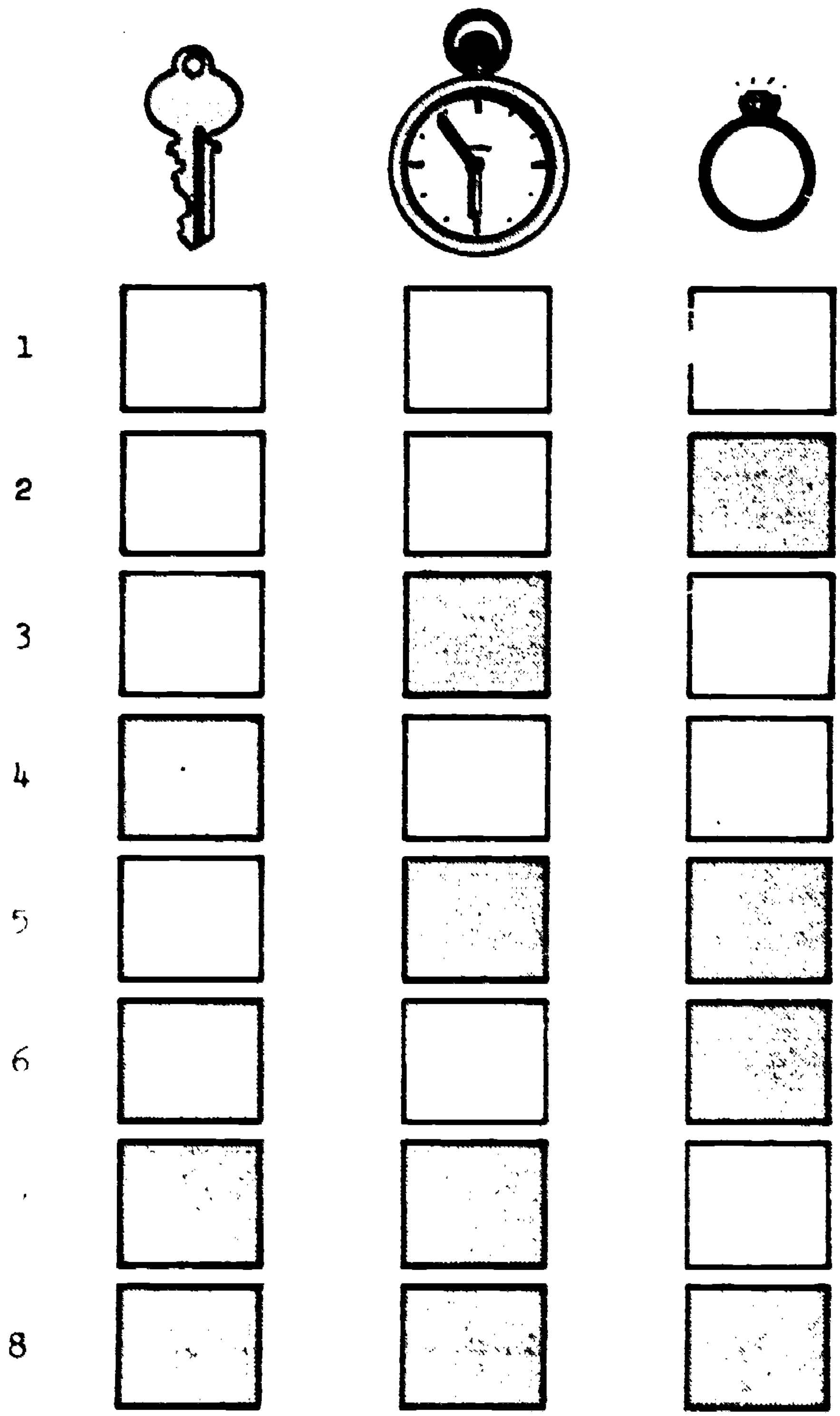
$$\frac{0}{1} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{1}{1} \quad \frac{3}{2} \quad \frac{2}{1} \quad \frac{3}{1} \quad \frac{1}{0}.$$

In this continued series every rational number will appear once and only once, and always in its simplest fractional form. There is no need, as there is in other methods of ordering the rationals, to eliminate fractions, such as 10/20, that are equivalent to simpler fractions also on the list, because no reducible fraction ever appears. If at each step you fill the cracks, so to speak, from left to right, you can count the fractions simply by taking them in their order of appearance. This series, as Peirce said, has many curious properties. At each new step the digits above the lines, taken from left to right, begin by repeating the top digits of the previous step: 01, 011, 0112 and so on. And at each step the digits below the lines are the same as those above the lines but in reverse order. The series is closely related to what are called Farey numbers (after the English geologist John Farey, who first analyzed them), about which there is now a considerable literature.

It is easy to show that there is a set with a higher infinite number of elements than aleph-null. To explain one of the best of such proofs a deck of cards is useful. First consider a finite set of three objects, say a key, a watch and a ring. Each subset of this set is symbolized by a row of three cards [*see illustration on opposite page*]; a face-up card [*white*] indicates that the object above it is in the subset, a face-down card [*gray*] indicates that it is not. The first subset consists of the original set itself. The next three rows indicate subsets that contain only two of the objects. They are followed by the three subsets of single objects and finally by the empty (or null) subset that contains none of the objects. For any set of n elements the number of subsets is 2^n. Note that this formula applies even to the empty set, since $2^0 = 1$ and the empty set has the empty set as its sole subset.

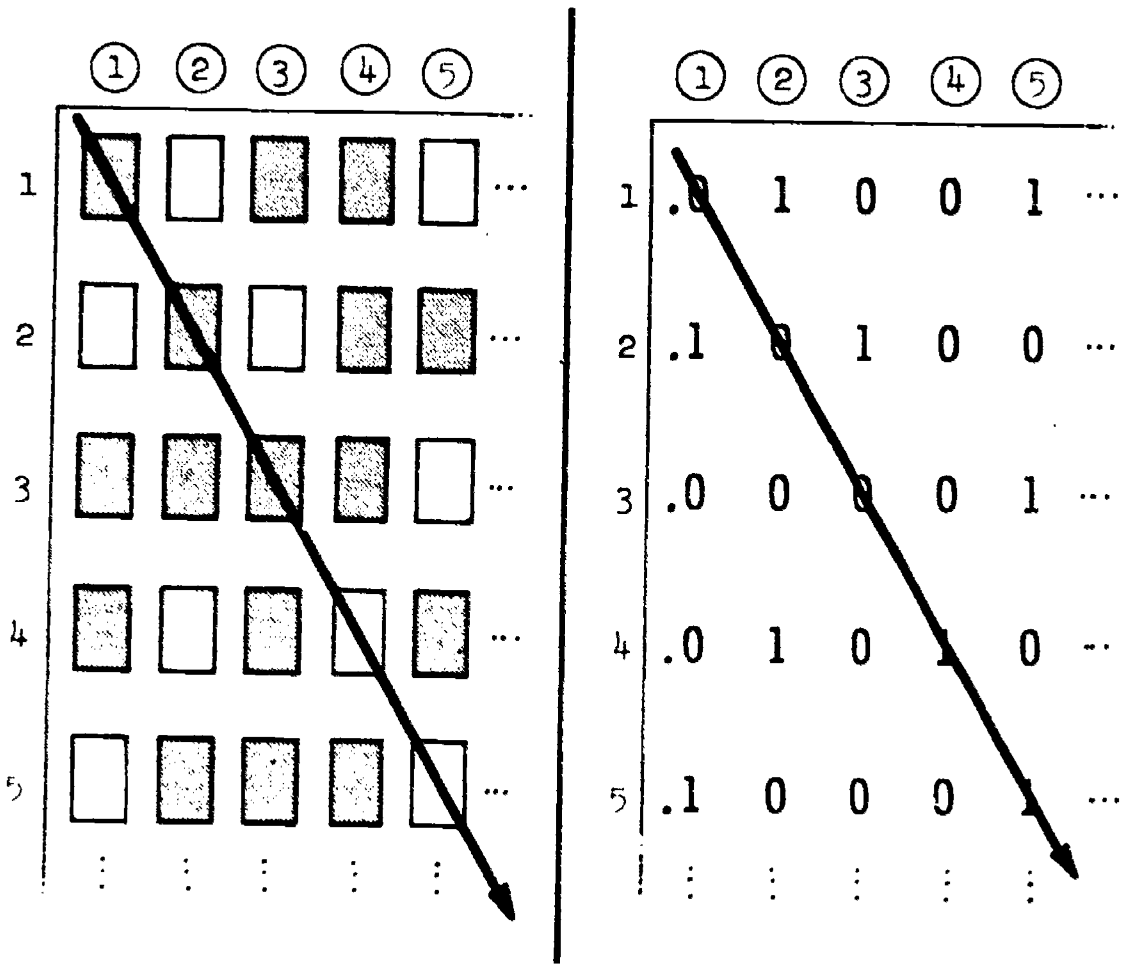
This procedure is applied to an infinite but countable (aleph-null) set of elements at the left in the illustration [*page 38*]. Can the subsets of this infinite set be put into one-to-one correspondence with the counting integers? Assume that they can. Symbolize each subset with a row of cards, as before, only now each row continues endlessly to the right. Imagine these infinite rows listed in any order whatever and numbered 1, 2, 3 . . . from the top down. If we continue forming such rows, will the list eventually catch all the subsets? No—because there is an infinite number of ways to produce a subset that cannot be on the list. The simplest way is to consider the diagonal set of cards indicated by the arrow and then suppose every card along this diagonal is turned over (that is, every face-down card is turned up, every face-up card is turned down). The new diagonal set cannot be the first subset because its first card differs from the first card of subset 1. It cannot be the second subset because its second card differs from the second card of subset 2. In general it cannot be the nth subset because its nth card differs from the nth card of subset n. Since we have produced a subset that cannot be on the list, even when the list is infinite, we are forced to conclude that the original assumption is false. The set of all subsets of an aleph-null set is a set with the cardinal number 2 raised to the power of aleph-null. This proof shows that such a set cannot be matched one to one with the counting integers. It is a higher aleph, an "uncountable" infinity.

Cantor's famous diagonal proof, in the form just given, conceals a startling bonus. It proves that the set of real numbers (the rationals



Subsets of a set of three elements

plus the irrationals) is also uncountable. Consider a line segment, its ends numbered 0 and 1. Every rational fraction from 0 to 1 corresponds to a point on this line. Between any two rational points there is an infinity of other rational points; nevertheless, even after all rational points are identified, there remains an infinity of unidentified points—points that correspond to the unrepeating decimal fractions attached to such algebraic irrationals as the square root of 2, and to such transcendental irrationals as pi and *e*. Every point on the line segment, rational or irrational, can be represented by an endless decimal fraction. But these fractions need not be decimal; they can also be written in binary notation. Thus every point on the line segment can be represented by an endless pattern of 1's and 0's, and every possible endless pattern of 1's and 0's corresponds to exactly one point on the line segment.



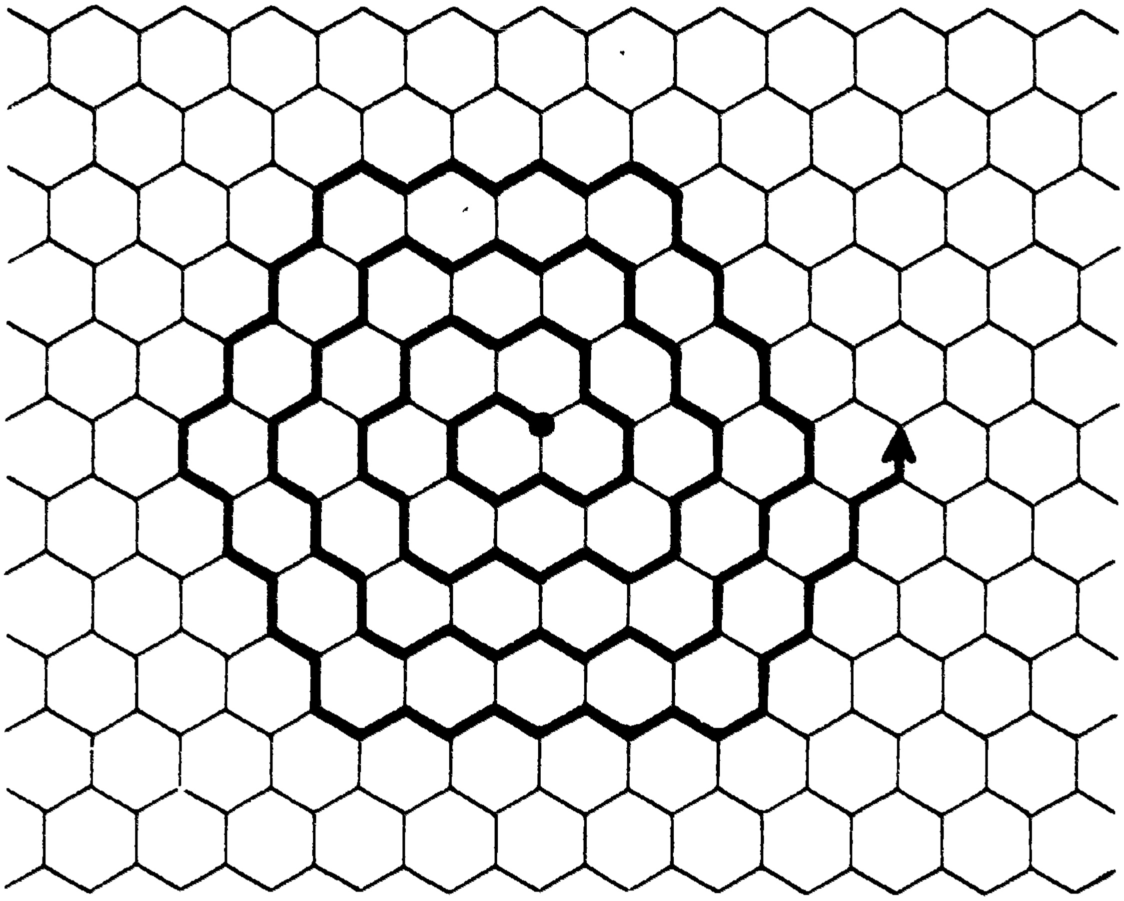
A COUNTABLE INFINITY HAS AN UNCOUNTABLE INFINITY OF SUBSETS (LEFT) THAT CORRESPOND TO THE REAL NUMBERS (RIGHT)

Now, suppose each face-up card at the left in the illustration on page 39 is replaced by 1 and each face-down card by 0, as shown at the right in the illustration. We have only to put a binary point in front of each row and we have an infinite list of different binary fractions between 0 and 1. But the diagonal set of symbols, after each 1 is changed to 0 and each 0 to 1, is a binary fraction that cannot be on the list. From this we see that there is a one-to-one correspondence of three sets: the subsets of aleph-null, the real members (here represented by binary fractions) and the totality of points on a line segment. Cantor gave this higher infinity the cardinal number C, for the "power of the continuum." He believed it was also $\aleph_1$ (aleph-one), the first infinity greater than aleph-null.

By a variety of simple, elegant proofs Cantor showed that C was the number of such infinite sets as the transcendental irrationals (the algebraic irrationals, he proved, form a countable set), the number of points on a line of infinite length, the number of points on any plane figure or on the infinite plane, and the number of points in any solid figure or in all of three-space. Going into higher dimensions does not increase the number of points. The points on a line segment one inch long can be matched one to one with the points in any higher-dimensional solid, or with the points in the entire space of any higher dimension.

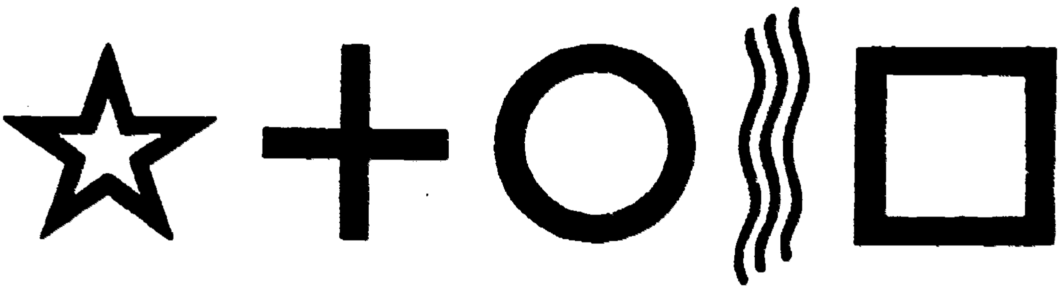
The distinction between aleph-null and aleph-one (we accept, for the moment, Cantor's identification of aleph-one with C) is important in geometry whenever infinite sets of figures are encountered. Imagine an infinite plane tessellated with hexagons. Is the total number of vertices aleph-one or aleph-null? The answer is aleph-null; they are easily counted along a spiral path [*see illustration on page 41*]. On the other hand, the number of different circles of one-inch radius that can be placed on a sheet of typewriter paper is aleph-one because inside any small square near the center of the sheet there are aleph-one points, each the center of a different circle with a one-inch radius.

Consider in turn each of the five symbols J. B. Rhine uses in his "ESP" test cards [*page 42*]. Can it be drawn an aleph-one number of times on a sheet of paper, assuming that the symbol is drawn with ideal lines of no thickness and that there is no overlap or intersection of any lines? (The drawn symbols need not be the same size, but all must be similar in shape.) It turns out that all except one can be drawn an aleph-one number of times. Can the reader show, before the answer is given, which symbol is the exception?



SPIRAL COUNTS THE VERTICES OF A HEXAGONAL TESSELLATION

The two alephs are also involved in recent cosmological speculation. Richard Schlegel, a physicist at Michigan State University, has called attention in several papers to a strange contradiction inherent in the "steady state" theory. According to that theory, the number of atoms in the cosmos at the present time is aleph-null. (The cosmos is regarded as infinite even though an "optical horizon" puts a limit on what can be seen.) Moreover, atoms are steadily increasing in number as the universe expands. Infinite space can easily accommodate any finite number of doublings of the quantity of atoms, for whenever aleph-null is multiplied by two, the result is aleph-null again. (If you have an aleph-null number of eggs in aleph-null boxes, one egg per box, you can accommodate another aleph-null set of eggs simply by shifting the egg in box 1 to box 2, the egg in box 2 to box 4, and so on, each egg going to a box whose number is twice the number of the egg's previous box. This empties all the odd-numbered boxes, which can then be filled with another aleph-null set of eggs.) But if the doubling goes on for an aleph-null number of times, we come up against the formula of 2 raised to the power of aleph-null — that is, $2 \times 2 \times 2 \ldots$ repeated aleph-null times. As we have seen, this produces an aleph-one set. Consider only two atoms at an infinitely remote time in the past. By now, after an aleph-null series of doublings, they would have grown to an aleph-one set. But the cosmos, at the moment, cannot contain an aleph-one set of atoms. Any collection of distinct physical entities (as opposed to the ideal entities of mathematics) is countable and therefore, *at the most*, aleph-null.



FIVE "ESP" SYMBOLS

In his latest paper, "The Problem of Infinite Matter in Steady-State Cosmology" (*Philosophy of Science*, Vol. 32, January, 1965, pages 21–31), Schlegel finds a clever way out. Instead of regarding the past as a completed aleph-null set of finite time intervals (to be sure, ideal instants in time form an aleph-one continuum, but Schlegel is concerned with those finite time intervals during which doublings of atoms occur), we can view both the past and the future as infinite in the inferior sense of "becoming" rather than completed. Whatever date is suggested for the origin of the universe (remember, we are dealing with the steady-state model, not with a "big bang" or oscillating theory), we can always set an earlier date. In a sense there is a "beginning," but we can push it as far back as we please. There is also an "end," but we can push it as far forward as we please. As we go back in time, continually halving the number of atoms, we never halve them more than a finite number of times, with the result that their number never shrinks to less than aleph-null. As we go forward in time, doubling the number of atoms, we never double more than a finite number of times; therefore the set of atoms never grows larger than aleph-null. In either direction the leap is never made to a completed aleph-null set of time intervals. As a result the set of atoms never leaps to aleph-one and the disturbing contradiction does not arise.

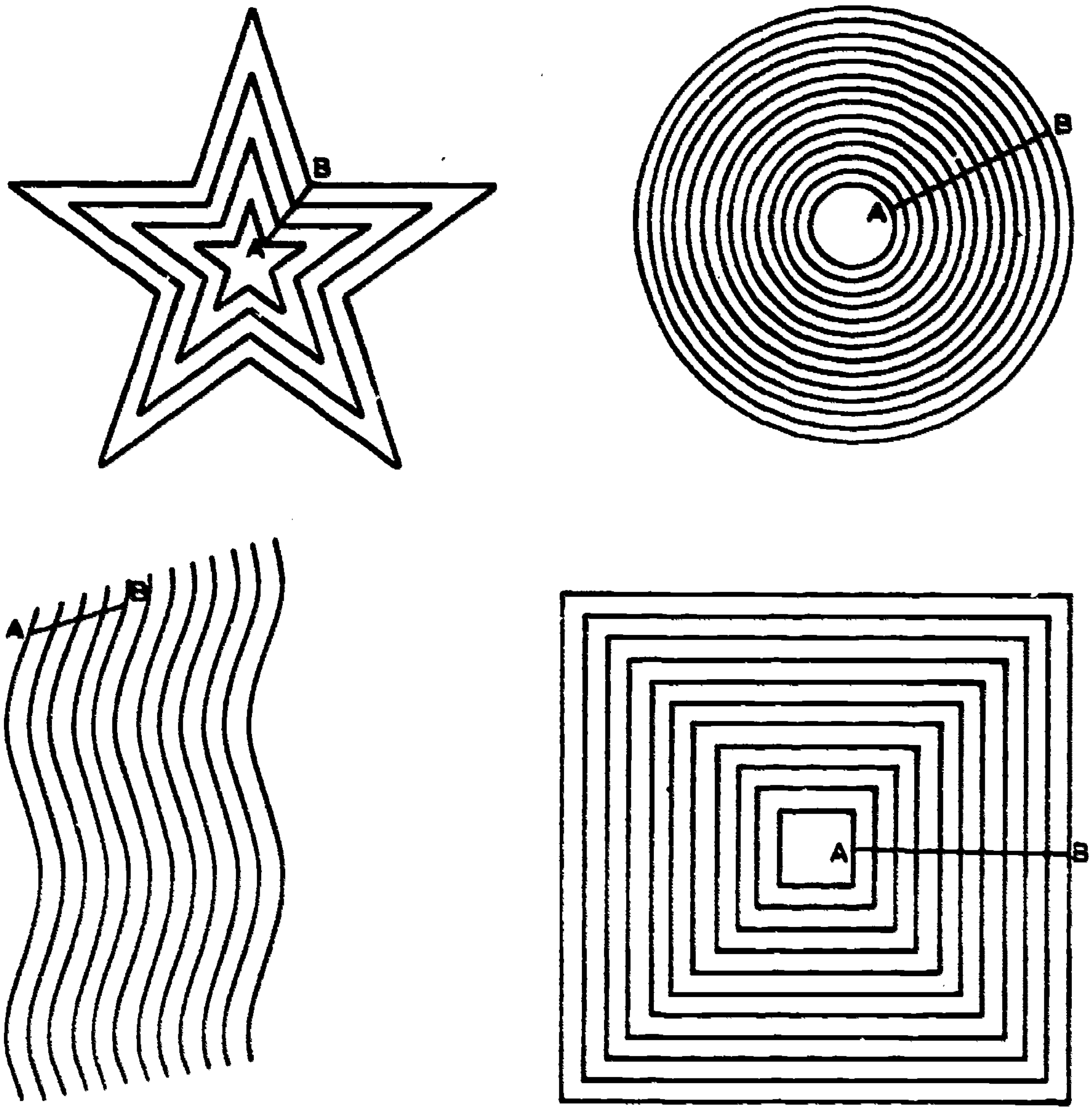
Cantor was convinced that his endless hierarchy of alephs, each obtained by raising 2 to the power of the preceding aleph, represented all the alephs there are. There are none in between. Nor is there an Ultimate Aleph, such as certain Hegelian philosophers of the time identified with the Absolute. The endless hierarchy of infinities itself, Cantor argued, is a better symbol of the Absolute.

All his life Cantor tried to prove that there is no aleph between aleph-null and *C*, the power of the continuum, but he never found a proof. In 1938 Kurt Gödel showed that Cantor's conjecture, which became known as the "continuum hypothesis," could be assumed to be true, and that this could not conflict with the axioms of set theory.

What Cohen proved in 1963 was that the opposite could also be assumed. One can posit that *C* is *not* aleph-one; that there is at least one aleph between aleph-null and *C*, even though no one has the slightest notion of how to specify a set (for example a certain subset of the transcendental numbers) that would have such a cardinal number. This too is consistent with set theory. Cantor's hypothesis is undecidable. Like the parallel postulate of Euclidean geometry, it is an independent axiom that can be affirmed or denied. Just as the two assumptions about Euclid's parallel axiom divided geometry into Euclidean and non-Euclidean, so the two assumptions about Cantor's hypothesis now divide the theory of infinite sets into Cantorian and non-Cantorian. Set theory has been struck a gigantic blow with a cleaver, and exactly what will come of it no one can say.

Last month's problem was to determine which of the five "ESP" symbols cannot be drawn an aleph-one number of times on a sheet of paper, assuming ideal lines that do not overlap or intersect, and replicas that are similar although not necessarily the same size. Only the plus symbol is limited to aleph-null repetitions. The illustration [*page 44*] shows how each of the other four can be drawn an aleph-one number of times. In each case points on line segment *AB* form an aleph-one continuum. Clearly a set of nested or side-by-side figures can be drawn so that a different replica passes through each of these points, thus putting the continuum of points into one-to-one correspondence with a set of nonintersecting replicas. There is no comparable way to place replicas of the plus symbol so that they fit snugly against each other. The centers of any pair of crosses must be a finite distance apart (although this distance can be made as small as one pleases), forming a countable (aleph-null) set of points.

The problem is similar to one involving alphabet letters that can be found in Leo Zippin's *Uses of Infinity* (Random House, 1962), page 57. In general only figures topologically equivalent to a line segment or a simple closed curve can be replicated on a plane, without intersection, aleph-one times.



PROOF FOR "ESP"-SYMBOL PROBLEM

FOREWORD

One of the most significant contributions of the ancient Greeks was the abiding faith in the worth of deductive reasoning, with the tacit assumption that deductive reasoning, suitably safeguarded, would never lead to contradictions. This faith has been sustained for over two thousand years. Yet hardly had Greek mathematics come into prominence when, ironically, doubts began to appear in the form of the sophistries proposed by Zeno (5th century B.C.), or, as they are frequently called, Zeno's paradoxes. These sophistries were logical arguments, which, although they led to disturbing conclusions, could not be refuted.

Zeno offered four paradoxes: (1) the *Dichotomy;* (2) *Achilles and the Tortoise;* (3) the *Arrow;* (4) the *Stade.* In two of these paradoxes he argued against the infinite divisibility of time and space. In the other two he showed that if a finite space and time contained only a finite number of points and instants, respectively, then we can arrive at conclusions which are contrary to experience.

Zeno's approach to the problem of the infinite may be summed up in his own words:[1]

> If there are many, they must be just so many as they are, neither more nor fewer. But if they are just so many as they are, they must be finite (in number).
>
> If there are many, the existents are infinite (in number); for there are always other (existents) between existents, and again others between these. And thus the existents are infinite (in number).

Curiously enough, Zeno's paradoxes have stirred controversies among mathematicians during the two thousand years that followed. It was not until Cantor created the continuum and the theory of aggregates (sets) that the paradoxes could finally be explained satisfactorily. So we conclude this collection of articles with N. Altshiller Court's lucid essay on *The Motionless Arrow* because this paradox is a classic; we place it last because, hopefully, it will be more meaningful to the reader *after* he has become familiar with the transfinite numbers of the twentieth century. Historical inversion sometimes has its advantages.

[1] Walter Kaufmann (Ed.). *Philosophic Classics*, Thales to St. Thomas. © 1961. By permission of Prentice-Hall, Inc., Englewood Cliffs, New Jersey. Also: *Evolution in Mathematical Thought* by Herbert Meschkowski, published by Holden-Day, Inc. 1965. Page 27. By permission.

The Motionless Arrow

N. A. COURT

In the lore of mankind the arrow occupies a conspicuous place, a place of distinction. There is the heroic arrow with which the legendary William Tell, at the behest of a tyrant, shot an apple off his own son's head, to say nothing of the other arrow that Tell held in reserve for the tyrant himself, in case his first aim should prove too low. There is the soaring first arrow of Hiawatha that would not touch the ground before the tenth was up in the air. There is the universally famous romantic arrow with which Cupid pierces the hearts of his favorites — or shall I say victims?

There is also an arrow that is philosophical, or scientific, or, better still, both. This famous "motionless arrow," as it may best be called, has stirred the mind, excited the imagination, and sharpened the wits of profound thinkers and erudite scholars for well over two thousand years.

Zeno of Elea, who flourished in the fifth century B.C., confronted his fellow philosophers and anybody else who was willing to listen with the bold assertion that an arrow, the swiftest object known to his contemporaries, cannot move at all.

According to Aristotle, Zeno's argument for, or proof of, his embarrassing proposition ran as follows: "Everything, when in uniform state, is continually either at rest or in motion, and a body moving in space is continually in the Now [instant], hence the arrow in flight is at rest." Some six centuries later another Greek philosopher offered a somewhat clearer formulation of the argument: "That which moves can neither move in the place where it is, nor yet in the place where it is not." Therefore, motion is impossible.

The "motionless arrow" was not Zeno's only argument of its kind. He had others. Zeno had Achilles engage in a race with a tortoise and showed a priori that the "light-of-foot" Achilles could never overtake the proverbially slow turtle. In Aristotle's presentation, here is the argument: "In a race the faster cannot overtake the slower, for the pursuer must always first arrive at the point from which the one pursued has just departed, so that the slower is always a small distance ahead." A modern philosopher states the argument more explicitly: "Achilles must first reach the place from which the tortoise has started. By that time the tortoise will have got on a little way. Achilles must then traverse that, and still the toroise will be ahead. He is always nearer, but he never makes up to it."

A third argument of Zeno's against motion is known as the "Dichotomy." In Aristotle's words: "A thing moving in space must arrive at the mid-point before it reaches the end-point." J. Burnet offers a more elaborate presentation of this argument:

> You cannot traverse an infinite number of points in a finite time. You must traverse half a given distance before you traverse the whole, and half of that again before you traverse it. This goes on ad infinitum, so that (if space is made up of points) there are an infinite number in any given space, and it cannot be traversed in a finite time.

Zeno had still other arguments of this kind. But I shall refrain from quoting them, for by now a goodly number of you have no doubt already begun to wonder what this is all about, what it is supposed to mean, if anything, and how seriously it is to be taken. Your incredulity, your skepticism, reflect the intellectual climate in which you were brought up and in which you continue to live. But that climate has not always been the same. It has changed more than once since the days of Zeno.

To take a simple example. We teach our children in our schools that the earth is round, that it rotates about its axis, and also that it revolves around the sun. These ideas are an integral part of our intellectual equipment, and it seems to us impossible to get along without them, much less to doubt them. And yet when Copernicus, or Mikolaj Kopernik, as the Poles call him, published his epoch-making work barely four centuries ago, in 1543, the book was banned as sinful. Half a century later, in 1600, Giordano Bruno was burned at the stake in a public place in Rome for adhering to the Copernican theory and other heresies. Galileo, one of the founders of modern science, for professing the same theories, was in jail not much more than three centuries ago.

What Zeno himself thought of his arguments, for what reason he advanced them, what purpose he wanted to achieve by them, cannot be told with any degree of certainty. The data concerning his life are scant and unreliable. None of his writings are extant. Like the title characters of some modern novels such as *Rebecca*, by Daphne du

Maurier, or *Mr. Skeffington*, by Elizabeth Arnim Russell, Zeno is known only by what is told of him by others, chiefly his critics and detractors. The exact meaning of his arguments is not always certain.

Zeno may or may not have been misinterpreted. But he certainly has not been neglected. Some writers even paid him the highest possible compliment — they tried to imitate him. Thus the "Dichotomy" suggested to Giuseppe Biancani, of Bologna, in 1615 a "proof" that no two lines can have a common measure. For the common measure, before it could be applied to the whole line, must first be applied to half the line and so on. Thus the measure cannot be applied to either line, which proves that two lines are always incommensurable.

A fellow Greek, Sextus Empiricus, of the third century A.D., taking the "motionless arrow" for his model, argued that a man can never die, for if a man die, it must be either at a time when he is alive or when he is dead, etc.

It may be of interest to mention in this connection that the Chinese philosopher Hui Tzu argued that a motherless colt never had a mother. When it had a mother it was not motherless and at every other moment of its life it had no mother.

Some writers offered very elaborate interpretations of Zeno's arguments. These writers saw in the creator of these arguments a man of profound philosophical insight and a logician of the first magnitude. Such was the attitude of Immanuel Kant and, a century later, of the French mathematician Jules Tannery. To Aristotle, who was born about a century after Zeno, these arguments were just annoying sophisms whose hidden fallacy it was all the more necessary to expose in view of the plausible logical form in which they were clothed. Other writers displayed just as much zeal in showing that Zeno's arguments are irrefutable.

Aristotle's fundamental assumptions are that both time and space are continuous, that is, "always divisible into divisible parts." He further adds: "The continual bisection of a quantity is unlimited, so that the unlimited exists potentially, but it is never reached."

With regard to the "Arrow" he says:

> A thing is at rest when it is unchanged in the Now and still in another Now, itself as well as its parts remaining in the same status.... There is no motion, nor rest in the Now.... In a time interval, on the contrary, it [a variable] cannot exist in the same state of rest, for otherwise it would follow that the thing in motion is at rest.

That it is impossible to traverse an unlimited number of half-distances (the "Dichotomy"), Aristotle refutes by pointing out that "time has unlimitedly many parts, in consequence of which there is no absurdity in the consideration that in an unlimited number of time intervals one passes over unlimited many spaces."

The argument Aristotle directs against "Achilles" is as follows:

If time is continuous, so is distance, for in half the time a thing passes over half the distance, and, in general, in the smaller time the smaller distance, for time and distance have the same divisions, and if one of the two is unlimited, so is the other. For that reason the argument of Zeno assumes an untruth, that one unlimited cannot travel over another unlimited along its own parts, or touch such an unlimited, in a finite time; for length as well as time and, in general, everything continuous, may be considered unlimited in a double sense, namely according to the [number of] divisions or according to the [distances between the] outermost ends.

Aristotle seems to insist that as the distances between Achilles and the tortoise keep on diminishing, the intervals of time necessary to cover these distances also diminish, and in the same proportion.

The reasonings of Aristotle cut no ice whatever with the French philosopher Pierre Bayle, who in 1696 published his *Dictionnaire Historique et Critique*, translated into English in 1710. Bayle goes into a detailed discussion of Zeno's arguments and is entirely on the side of Zeno. He categorically rejects the infinite divisibility of time.

Successive duration of things is composed of moments, properly so called, each of which is simple and indivisible, perfectly distinct from time past and future and contains no more than the present time. Those who deny this consequence must be given up to their stupidity, or their want of sincerity, or to the unsurmountable power of their prejudices.

Thus the "Arrow" will never budge.

The philosophical discussion of the divisibility or the nondivisibility of time and space continues through the centuries. As late as the close of the past century Zeno's arguments based on this ground were the topic of a very animated discussion in the philosophical journals of France.

A mathematical approach to "Achilles" is due to Gregory St. Vincent who in 1647 considered a segment AK on which he constructed an unlimited number of points $B, C, D, \ldots$ such that $AB/AK = BC/BK = CD/CK = \ldots = r$, where r is the ratio, say, of the speed of the tortoise to the speed of Achilles. He thus obtains the infinite geometric progression $AB + BC + CD + \ldots$, and, since this series is convergent, Achilles does overtake the elusive tortoise.

Decartes solved the "Achilles" by the use of the geometric progression $1/10 + 1/100 + 1/1000 + \ldots = 1/9$. Later writers quoted this device or rediscovered it time and again. But this solution of the problem raised brand-new questions.

St. Vincent overlooked the important fact that Achilles will fail to overtake the slow-moving tortoise after all, unless the variable sum of the geometric progression actually reaches its limit. Now: Does a variable reach its limit, or does it not? The question transcends, by far, the "Achilles." It was, for instance, hotly debated in connection with the then nascent differential and integral calculus. Newton believed that his variables reached their limits. Diderot, writing a century or so later in the famous *Encyclopédie*, is quite definite that a variable cannot do that, and so is De Morgan, in the *Penny Cyclopedia* in 1846. Carnot and Cauchy, like Newton, have no objections to variables reaching their limits.

The other question that arises in connection with St. Vincent's progression is: How many terms does the progression have? The answer ordinarily given is that the number is infinite. This answer, however, may have two different meanings. We may mean to say that we can compute as many terms of this progression as we want and, no matter how many we have computed, we can still continue the process. Thus the number of terms of the progression is "potentially" infinite. On the other hand, we may imagine that all the terms have been calculated and are all there forming an infinite collection. That would make an "actual" infinite. Are there actually infinite collections in nature? Obviously, collections as large "as the stars of the heaven, and as the sand which is upon the seashore," are nevertheless finite collections.

From a quotation of Aristole already given it would seem that he did not believe in the actually infinite. Galileo, on the other hand, accepted the existence of actual infinity, although he saw clearly the difficulties involved. If the number of integers is not only potentially but actually infinite. then there are as many perfect squares as there are integers, since for every integer there is a perfect square and every perfect square has a square root. Galileo tried to console himself by saying that the difficulties are due to the fact that our finite mind cannot cope with the infinite. But De Morgan sees no point to this argument, for, even admitting the "finitude" of our mind, "it is not necessary to have a blue mind to conceive of a pair of blue eyes."

A younger contemporary of Galileo, the prominent English philosopher Thomas Hobbes (1588–1679), could not accept Galileo's actual infinity, on theological grounds. "Who thinks that the number of even integers is equal to the number of all integers is taking away eternity from the Creator." However, the very same theological reasons led a very illustrious younger contemporary of Hobbes, namely, Leibnitz, to the firm belief that actual infinities exist in nature *pour mieux marquer les perfections de son auteur.*

The actual infinite was erected into a body of doctrine by Georg Cantor (1845–1918) in his theory of transfinite numbers. The outstanding American historian of mathematics, Florian Cajori, considers that this doctrine of Cantor's provided a final and definite answer to Zeno's paradoxes and thus relegates them to the status of "problems of the past."

Tobias Danzig in his *Number, the Language of Science* is not quite so happy about it, in view of the fact that the whole theory of Cantor's is of doubtful solidity.

Whatever may have been the reasons that prompted Zeno to promulgate his paradoxes, he certainly must have been a man of courage if he dared to deny the existence of motion. We learn of motion and learn to appreciate it at a very, very early age; motion is firmly imbedded in our daily existence and becomes a basic element of our psychological make-up. It seems intolerable to us that we could be deprived of motion, even in a jest.

Nevertheless, the systematic study of motion is of fairly recent origin. The ancient world knew a good deal about Statics, as evidenced by the size and solidity of the structures that have survived to the present day. But they knew next to nothing about Dynamics, for the forms of motion with which they had any experience were of very narrow scope. Their machines were of the crudest and very limited in variety. Zeno's paradoxes of motion were for the Greek philosophers "purely academic" questions.

The astronomers were the first to make systematic observations of motion not due to muscular force and to make deductions from their observations. Man studied motion in the skies before he busied himself with such studies on earth. How difficult it was for the ancients to dissociate motion from muscular effort is illustrated by the fact that Helios (the sun) was said by the Greeks to have a palace in the east whence he was drawn daily across the sky in a fiery chariot by four white horses to a palace in the west.

The famous experiments of Galileo with falling bodies are the beginning of modern Dynamics. The great voyages created a demand for reliable clocks, and the study of clock mechanisms and their motion engaged the attention of such outstanding scholars as Huygens. No small incentive for the study of motion was provided by the needs of the developing artillery. The gunners had to know the trajectories of their missiles. The theoretical studies of motion prompted by these and other technical developments were in need of a new mathematical tool to solve the newly arising problems, and calculus came into being.

The infinite, the infinitesimal, limits and other notions that were involved, perhaps crudely, in the discussion of Zeno's arguments were also involved in this new branch of mathematics. These notions were as hazy as they were essential. Both Newton and Leibnitz changed their views on these points during their lifetimes because of their own critical acumen as well as the searching criticism of their contemporaries. But neither of them ever entertained the idea of giving up their precious find, for the good and sufficient reason that this new and marvelous tool gave them the solution of some of the problems that had defied all the efforts of mathematicians of preceding generations. The succeeding century, the eighteenth, exploited to the utmost this new instrument in its application to the study of motion, and before the century was over it triumphantly presented to the learned world two monumental works: the *Mécanique Analytique* of Lagrange and the *Mécanique Céleste* of Laplace.

The development of Dynamics did not stop there. It kept pace with the phenomenal development of the experimental sciences in the nineteenth century. These theoretical studies on the one hand served as a basis for the creation of a technology that surpassed the wildest dreams of past generations and on the other hand changed radically our attitude toward many of the problems of the past; they created a new intellectual atmosphere, a new "intellectual climate."

Zeno's arguments, or paradoxes, if you prefer, deal with two questions which in the discussions of these paradoxes are very closely connected, not to say mixed up: What is motion, and how can motion be accounted for in a rational, intellectual way? By separating the two parts of the problem we may be able to come much closer to finding a satisfactory answer to the question, in accord with the present-day intellectual outlook.

The critical study of the foundations of mathematics during the nineteenth century made it abundantly clear that no science and, more generally, no intellectual discipline can define all the terms it uses without creating a vicious circle. To define a term means to reduce it to some more familiar component parts. Such a procedure obviously has a limit beyond which it cannot go. Most of us know what the color "red" is. We can discuss this color with each other; we can wonder how much the red color contributes to the beauty of a sunset; we can make use of this common knowledge of the red color for a common purpose, such as directing traffic. But we cannot undertake to explain what the red color is to a person born color blind.

In the science of Dynamics motion is such a term, such an "undefined" term, to use the technical expression for it. Dynamics does not propose to explain what motion is to anyone who does not know that already. Motion is one of its starting points, one of its undefined, or primitive, terms. This is its answer to the question: What is motion?

You have heard many stories about Diogenes. He lived in a barrel. He threw away his drinking cup when he noticed a boy drinking out of the hollow of his hand. He told his visitor, Alexander the Great, that the only favor the mighty conqueror could possibly do him was to step aside so as not to obstruct the sun for the philosopher. Well, there is also the story that when Diogenes was told of Zeno's arguments about the impossibility of motion, he arose from the place where he was sitting on the ground alongside his barrel, took a few steps, and returned to his place at the barrel without saying a single word. This was the celebrated Cynic philosopher's "eloquent" way of saying that motion *is*. And did he not also say at the same time that motion is an "undefined term"?

St. Augustine (354–430) used an even more convincing method to emphasize the same point. He wrote:

> When the discourse [on motion] was concluded, a boy came running from the house to call for dinner. I then remarked that this boy compels us not only to *define* motion, but to see it before our very eyes. So let us go and pass from one place to another, for that is, if I am not mistaken, nothing else than motion.

The revered theologian seems to have known, from personal experience, that nothing is as likely to set a man in motion as a well-garnished table.

Let us now turn to the second part involved in Zeno's paradoxes, namely, how to account for motion in a rational way. All science may be said to be an attempt to give a rational account of events in nature, of the ways natural phenomena run their courses. The scientific theories are a rational description of nature that enables us to foresee and foretell the course of natural events. This characteristic of scientific theories affords us an intellectual satisfaction, on the one hand, and, on the other hand, shows us how to control nature for our benefit, to serve our needs and comforts. *Prévoir pour pouvoir,* to quote Henri Poincaré. A scientific theory, that is, a rational description of a sector of nature, is acceptable and accepted only as long as its previsions agree with the facts of observation. There can be no bad theory. If a theory is bad or goes bad, it is modified or it is thrown out completely.

"Achilles" is an attempt at a rational account of a race, a theoretical interpretation of a physical phenomenon. The terrible thing is that Zeno's theory predicts one result, while everybody in his senses knows quite well that exactly the contrary actually takes place. Aristotle in his time and day felt called upon to use all his vast intellectual powers to refute the paradox. Our present intellectual climate imposes no such obligation upon us. If saying that in order to overtake the tortoise Achilles must first arrive at the point from which the tortoise started, etc., leads to the conclusion that he will never overtake the creeping animal, we simply infer that Zeno's theory of a race does not serve the purpose for which it was created. We declare the scheme to be unworkable and proceed to evolve another theory which will render a more satisfactory account of the outcome of the race.

That, of course, is assuming that the theory of Zeno was offered in good faith. If it was not, then it is an idle plaything, very amusing, perhaps, very ingenious, if you like, but not worthy of any serious consideration. There are more worthwhile ways of spending one's time than in shadow boxing. Our indifferent attitude towards Zeno's paradoxes is perhaps best manifested by the fact that the article "Motion" in the *Britannica* does not mention Zeno, whereas Einstein is given considerable attention; the *Americana* dismisses "Motion" with the curt reference "see Mechanics."

Consider an elastic ball which rebounds from the ground to $\frac{2}{3}$ of the height from which it fell. When dropped from a height of 30 feet, how far will the ball have traveled by the time it stops? Any bright freshman will immediately raise the question whether that ball will ever stop. On the other hand, that same freshman knows full well that after a while the ball will quietly lie on the ground. Will we be very much worried by this contradiction? Not at all. We will simply draw the conclusion that the law of rebounding of the ball, as described, is faulty.

The difficulties encountered in connection with the question of a variable reaching or not reaching a limit are of the same kind and nature. The mode of variation of a variable is either a description of a natural event or a creation of our imagination, without any physical connotation. In the latter case, the law of variation of the variable is prescribed by our fancy, and the variable is completely at our mercy. We can make it reach the limit or keep it from doing so, as we may see fit. In the former case it is the physical phenomenon that decides the question for us.

Two bicycle riders, 60 miles apart, start towards each other, at the rate of 10 miles per hour. At the moment when they start a fly takes off from the rim of the wheel of one rider and flies directly towards the second rider at the rate of 15 miles per hour. As soon as the fly reaches the second rider it turns around and flies towards the first, etc. What is the sum of the distances of the oscillations of the fly? In Zeno's presentation the number of these oscillations is infinite. But the flying time was exactly 3 hours, and the fly covered a distance of 45 miles. The variable sum actually reached its limit.

The sequence of numbers 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, . . . obviously has for its limit zero. Does the sequence reach its limit? Let us interpret this sequence, somewhat facetiously, in the following manner. A rabbit hiding in a hollow log noticed a dog standing at the end near him. The rabbit got scared and with one leap was at the other end; but there was another dog. The rabbit got twice as scared, and in half the time he was back at the first end; but there was the first dog, so the rabbit got twice as scared again, etc. If this sequence reaches its limit, the rabbit will end up by being at both ends at the same time.

If a point Q of a curve (C) moves towards a fixed point P of the curve, the line PQ revolves about P. If Q approaches P as a limit, the line PQ obviously approaches as a limiting position the tangent to the curve (C) at the point P; and if the point Q reaches the position P or, what is the same thing, coincides with P, the line PQ will coincide with the tangent to (C) at P.

If s represents the distance traveled by a moving point in the time t, does the ratio s/t approach a limit when t approaches zero as a limit? In other words, does a moving object have an instantaneous velocity at a point of its course, or its trajectory? Aristotle could not answer that question; he probably could not make any sense of the question. Aristotle agreed with Zeno that there can be no motion in the Now (moment). But to us the answer to this question is not subject to any doubt whatever: we are too accustomed to read the instantaneous velocities on the speedometers of our cars.

The divisibility or the nondivisibility of time and space was a vital question to the Greek philosophers, and they had no criterion according to which they could settle the dispute. To us time and space are constructs that we use to account for physical phenomena, constructs of our own making, and as such we are free to use them in any manner we see fit. Albert Einstein did not hesitate to mix the two up and make of them a space-time continuum when he found that such a construct is better adapted to account for physical phenomena according to his theory of relativity.

I have dealt with the two parts of Zeno's paradoxes: the definition of motion and the description of motion. There is, however, a third element in these paradoxes, and it is this third element that is probably more responsible for the interest that these paradoxes held throughout the centuries than those I have considered already. This is the logical element.

That Zeno was defending an indefensible cause was clear to all those who tried to refute him. But how is it possible to defend a false cause with apparently sound logic? This is a very serious challenge. If sound logic is not an absolute guaranty that the propositions defended by that method are valid, all our intellectual endeavors are built on quicksand, our courts of justice are meaningless pantomime, etc.

Aristotle considered that the fundamental difficulty involved in Zeno's argument against motion was the meaning Zeno attached to his "Now." If the "Now," the moment, as we would say, does not represent any length of time but only the durationless boundary between two adjacent intervals of time, as a point without length is the common boundary of two adjacent segments of a line, then in such a moment there can be no motion; the arrow is motionless. Aristotle tried to refute Zeno's denial of motion by pointing out that it is wrong to say that time is made up of durationless moments. But Aristotle was not very convincing, judging by the vitality of Zeno's arguments.

Our modern knowledge of motion provides us with better ways of meeting Zeno's paradoxes. We can grant Zeno both the durationless "Now" and the immobility of the object in the "Now" and still contend that these two premises do not imply the immobility of the arrow. While the arrow does not move in the "Now," it conserves its capacity, its potentiality of motion. In our modern terminology, in the "Now" the arrow has an instantaneous velocity. This notion of instantaneous velocity is commonplace with us; we read it "with our own eyes" on our speedometers every day. But it was completely foreign to the ancients. Thus Zeno's reasoning was faulty because he did not know enough about the subject he was reasoning about.

Zeno's apparently unextinguishable paradoxes, as they are referred to by E. T. Bell in an article recently published in *Scripta Mathematica,* will not be put out of circulation by my remarks about them. I have no illusions about that; neither do I have any such ambitions. These paradoxes have amused and excited countless generations, and they should continue to do so. Why not?

FOR FURTHER READING AND STUDY

BOLZANO, BERNARD. *Paradoxes of the Infinite*. Yale University Press, 1950. 189 p.

CANTOR, GEORG. *Contributions to the Founding of the Theory of Transfinite Numbers*. Dover Publications, 1915.

KASNER, EDWARD AND NEWMAN, JAMES. *Mathematics and the Imagination*. Simon & Schuster, 1940. Chapter 2: "Beyond the Googol."

KEYSER, C. J. *The Rational and the Superrational* (Collected Works, Vol. 2) Scripta Mathematica, 1952. Pp. 169–198: "The Role of Infinity in the Cosmology of Epicurus."

LIEBER, LILLIAN R. AND LIEBER, HUGH GRAY. *Infinity*. Rinehart and Company, 1953. 359 p.

MESCHKOWSKI, HERBERT. *Evolution of Mathematical Thought*. Holden-Day, Inc., 1965. Chapter 4: "The Problem of Infinity."

NORTHROP, EUGENE. *Riddles in Mathematics: A Book of Paradoxes*. Van Nostrand, 1944. Chapter 7: "Paradoxes of the Infinite."

PEDOE, DAN. *The Gentle Art of Mathematics*. English Universities Press, Ltd. (London), 1958. Chapter 3: "Where Does It End?"

PÉTER, RÓZSA. *Playing With Infinity*. G. Bell & Sons, Ltd. (London). 1961. Of especial interest: Sections 3, 10, 11, 12, and 21.

SINGH, JAGJIT. *Mathematical Ideas: Their Nature and Use*. Hutchinson & Co. (London), 1959. Chapter 5: "Zeno and Infinity."

STEIN, SHERMAN K. *Mathematics: The Man-Made Universe*. W. H. Freeman & Co., 1963. Chapter 16: "Infinite Sets."

TIETZE, HEINRICH. *Famous Problems of Mathematics*. Graylock Press, 1965. Chapter 12: "Infinity in Mathematics."

YARNELLE, JOHN E. *An Introduction to Transfinite Mathematics*. D. C. Heath & Co., 1964. (Paper). 65 p.

YOUNG, FREDERICK H. *The Nature of the Regular Polyhedra – Infinity and Beyond – An Introduction to Groups*. Ginn and Company, 1961. (Pamphlet) Pp. 9–15.

ZIPPIN, LEO. *Uses of Infinity*. L. W. Singer (Random House), 1962. 151 p.

—W.L.S.